An Introduction to Modular Forms and the Eichler-Shimura Isomorphism

Thomas Andrew Lamb

Supervisor: Dr. Herbert Gangl

May 27, 2022

Abstract

There are two main aims for this project, the first is to give a complete and detailed account of the basic theory of classical modular forms for the group $SL_2(\mathbb{Z})$, the second is to state and prove the result know as the Eichler-Shimura isomorphism.

Contents

Chapter 1

Introduction

1.1 Motivation

Modular forms are holomorphic functions on the upper half plane that have infinitely many ‘symmetric’ properties. These functions contain important arithmetic information, which, when ‘teased’ out, leads to some rather remarkable results. Modular forms were first studied in connection to elliptic functions in the nineteenth century. Since then, mathematicians have become increasingly aware of just how fundamental these objects really are in modern mathematics. This is famously highlighted by the significant role they play in the proof of Fermat’s Last theorem [Wil95]. Moreover, these special functions seem to be pervasive throughout mathematics and naturally appear in areas as varied as mathematical physics and number theory; a nice survey of such appearances and applications can be found in [Zag08].

1.2 Content

In this project, we will be discussing classical modular forms with respect to the group $SL_2(\mathbb{Z})$. Our aim is two fold:

i) Give a full and detailed account of the basic theory of modular forms with respect to $SL_2(\mathbb{Z})$.

ii) Introduce an important result in the subject, namely, the Eichler-Shimura isomorphism.

The first four chapters of this project give an in depth account of the fundamental theory of modular forms for $SL_2(\mathbb{Z})$. Firstly, we will define what a modular form for $SL_2(\mathbb{Z})$ is and show that such functions form complex vector spaces. It will turn out that such spaces are finite-dimensional, and it is this important property that gives rise to many interesting results, some for which we will see in this project.

Next, we will discuss Hecke operators and Dirichlet series attached to modular forms. These two topics are very important. Hecke operators give us a powerful tool which we can use to study the spaces of modular forms, and Dirichlet series attached to 'Hecke eigenforms' will give us an interesting connection to number theory through Euler products.

In the remaining two chapters, we will work towards stating and proving the result know as the Eichler-Shimura isomorphism. This result links subspaces of modular forms to cohomology groups of $SL_2(\mathbb{Z})$ using L-functions and antiderivatives of modular forms. This significant isomorphism has connections to periods and period polynomials, which are intriguing subjects in their own right. Both period polynomials and the Eichler-Shimura isomorphism (in its various forms and generalisations) still draw a lot of attention from mathematicians, something we will briefly illustrate towards the end of this project.

This project is aimed at masters level mathematics students with a strong background in the fundamentals of complex analysis.

Chapter 2

The Modular Group

In this chapter we introduce the Modular Group Γ and describe its action on the upper half plane, $\mathbb{H}$. We then derive the famous fundamental domain for Γ and finish by showing that Γ can be generated by only two elements.The proofs are standard and can be found in most textbooks on the subject. Here, we will be following [Ser12, pp. 77–79].

2.1 The Modular Group and its Action on $\mathbb{H}$

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ denote the upper half plane. Define a left action of $SL_2(\mathbb{Z}) = \{\gamma \in M_2(\mathbb{Z}) \mid \det(\gamma) = 1\}$ on $\mathbb{C}$, denoted $SL_2(\mathbb{Z}) \circlearrowleft \mathbb{C}$, by

$$SL_2(\mathbb{Z}) \times \mathbb{C} \to \mathbb{C},\ (\gamma, z) \mapsto \gamma z = \frac{az+b}{cz+d}\ ,\ \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

This is indeed an action as

$$Iz = \frac{z+0}{0+1} = z,\ \text{for } z \in \mathbb{C},$$

and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\delta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, with $\gamma, \delta \in SL_2(\mathbb{Z})$, we have that

$$(\gamma\delta)z = \frac{(ae+bg)z + af + bh}{(ce+dg)z + cf + dh} = \left(a\frac{ez+f}{gz+h} + b\right) \Big/ \left(c\frac{ez+f}{gz+h} + d\right) = \gamma(\delta z),\ \text{for } z \in \mathbb{C}.$$

By restricting to $\mathbb{H}$, we also have an action $SL_2(\mathbb{Z}) \circlearrowleft \mathbb{H}$. This is because for $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have that

$$\operatorname{Im}(\gamma z) = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}\left(\frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)}\right) = \frac{\operatorname{Im}(z)(ad-bc)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2} > 0,$$

since $\operatorname{Im}(z) > 0$ and $\det(\gamma) = ad - bc = 1$.

Remark 2.1.1. Note that given $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have that

$$(-\gamma)z = \frac{-az - b}{-cz - d} = \frac{az + b}{cz + d} = \gamma z.$$

So $-\gamma$ gives the same action as γ.

$SL_2(\mathbb{Z})$ will play an important role in defining the modular forms we will be working with, so it makes sense to give it a name.

Definition 2.1.2 (The Modular Group)**.** Define the modular group as $\Gamma = SL_2(\mathbb{Z})$. [1]

2.2 The Fundamental Domain of the Modular Group

Our next goal for this chapter is to derive the fundamental domain for the action of Γ on $\mathbb{H}$.

As a reminder, for X a topological space, the closure of a set $\mathcal{F} \subset X$, denoted $\overline{\mathcal{F}}$, is the smallest closed set containing $\mathcal{F}$. The interior of $\mathcal{F}$, denoted $\mathring{\mathcal{F}}$, is the largest open set contained in $\mathcal{F}$. We define the boundary, denoted $\partial\mathcal{F}$, as $\partial\mathcal{F} = \overline{\mathcal{F}} \setminus \mathring{\mathcal{F}}$. For details and more precise notions, see [Arm13, p. 30].

Definition 2.2.1 (Fundamental Domain of a Group Action)**.** Let X be a topological space with $\mathcal{F} \subset X$, an open and connected subset (i.e. a domain). Let G be a group with a (left) action $G \circlearrowleft X$. We say $\mathcal{F}$ is a fundamental domain for this group action if we have that the sets $g\mathcal{F} = \{gz \mid z \in \mathcal{F} \text{ and } g \in G\}$ satisfy the following conditions:

1. $X = \bigcup_{g \in G} \overline{g\mathcal{F}}$.
2. For distinct $z, w \in \mathcal{F}$, we have that $w \notin Orb(z)$. [2]

This definition means that each orbit has a representative in $\overline{\mathcal{F}}$. Moreover, the orbits of distinct element in $\mathcal{F}$ are disjoint, i.e. they give distinct elements in the orbit space $G\backslash X$ (the set of orbits).

We will now derive the famous fundamental domain for Γ

Theorem 2.2.2. *The set $\mathcal{F} = \{z \in \mathbb{H} \mid 1 < |z| \text{ and } |\mathrm{Re}(z)| < 1/2\}$ is a fundamental domain of Γ.*

Note here that $\overline{\mathcal{F}} = \{z \in \mathbb{H} \mid 1 \leq |z| \text{ and } |\mathrm{Re}(z)| \leq 1/2\}$ and $\partial\mathcal{F} = \{e^{i\theta} \mid \pi/3 \leq \theta \leq 2\pi/3\} \cup \{z \in \mathbb{H} \mid \mathrm{Re}(z) = \pm 1/2 \text{ and } |z| > 1\}$.

[1] Some authors, e.g. Serre in [Ser12], define the modular group as $PSL_2(\mathbb{Z}) = \Gamma/\{\pm I\}$ as, with some work, one can show that $PSL_2(\mathbb{Z})$ acts faithfully on $\mathbb{H}$ (i.e. only the identity element of $PSL_2(\mathbb{Z})$ acts trivially), whereas Remark 2.1.1 shows that $SL_2(\mathbb{Z})$ does not.

[2] If we require the stronger condition that for every element $g \in G$, with g not the identity element, $g\mathcal{F} \cap \mathcal{F} = \emptyset$, then we see, considering the element $-I$, that $\mathcal{F}$, as defined in Theorem 2.2.2, is not the fundamental domain for $SL_2(\mathbb{Z})$. However, it would be the fundamental domain for $PSL_2(\mathbb{Z})$.

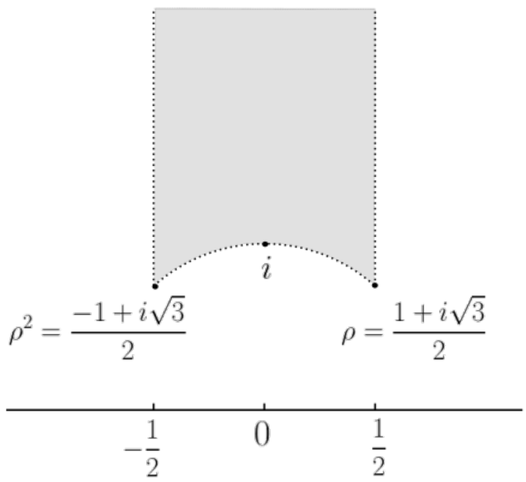

Figure 2.0: The fundamental domain $\mathcal{F}$ of the action $\Gamma \circlearrowleft \mathbb{H}$.

Proof. We will use the elements

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

to first show that any $z \in \mathbb{H}$ can be moved into $\overline{\mathcal{F}}$ by the repeated action of these two elements. Define $H = \langle R, T \rangle$, the subgroup of Γ generated by R, T and their inverses. The first matrix corresponds to the action $Rz = -1/z$, the second to $Tz = z + 1$. From this, we see that Rz is a composition of an inversion in the unit circle (given by $z \mapsto 1/\bar{z}$) with a reflection in the imaginary axis (given by $z \mapsto -\bar{z}$), and Tz is a translation parallel to the real axis by a unit.

Let $z \in \mathbb{H}$, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$. Since $\gamma \in H$, by definition, $\det(\gamma) = ad - bc = 1$. Hence, we cannot have both c and d zero. We know from our work in the previous section that

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im}(z)}{|cz + d|^2} . \tag{2.1}$$

Note that for fixed $z \in \mathbb{H}$, we may define the lattice $\Lambda_z = \mathbb{Z}z \oplus \mathbb{Z}$. Note that the elements in H generate a subset of Λ_z, by considering, for example in the case of γ above, $cz + d \in \Lambda_z$. Since a lattice is discrete in $\mathbb{C}$ (which refers to the fact we can isolate each point of the lattice with a small enough disc), any subset of a lattice is also discrete in $\mathbb{C}$. The size of the set formed from the intersection of a discrete set with any compact set (by Heine–Borel, in the case of $\mathbb{C}$, being compact is equivalent to being closed and bounded) is finite. Hence, there exists an element $\delta \in H$ such that $|cz + d|$ is minimised over all elements in H. Hence, (2.1) implies that

$$\operatorname{Im}(\gamma z) \leq \operatorname{Im}(\delta z), \quad \text{for all } \gamma \in H.$$

Given a $z \in \mathbb{H}$, we can find a $j \in \mathbb{Z}$ such that $T^j \delta z \in \{w \in \mathbb{H} \mid |\mathrm{Re}(w)| \leq \frac{1}{2}\}$, i.e. we can translate δz into the vertical strip of width one containing $\mathcal{F}$. If $T^j \delta z \in \overline{\mathcal{F}}$, then we are done. Else, we must have $\left|T^j \delta z\right| < 1$. We now apply S to $T^j \delta z$. This gives

$$\mathrm{Im}(S(T^j \delta z)) = \frac{\mathrm{Im}(T^j \delta z)}{|T^j \delta z|^2} > \mathrm{Im}(T^j \delta z) = \mathrm{Im}(\delta z).$$

But, $\mathrm{Im}(\delta z)$ is maximal. This gives a contradiction, meaning $\delta z \in \overline{\mathcal{F}}$. Hence, we have demonstrated the first condition of Definition 2.2.1 is satisfied (note, however, by only using elements of H).

We now move on to the second condition in Definition 2.2.1. Here, we will prove more than we actually need, but in fact this will help us in the next section to deduce an interesting fact about the group $SL_2(\mathbb{Z})$.

Suppose we have an element $\gamma \in \Gamma$ and further suppose we have a $z \in \overline{\mathcal{F}}$ with $\gamma z \in \overline{\mathcal{F}}$ as well. Assume without loss of generality that $\mathrm{Im}(\gamma z) \geq \mathrm{Im}(z)$. We can assume this since if we had that $\mathrm{Im}(z) \geq \mathrm{Im}(\gamma z)$, we could define $w = \gamma z \in \overline{\mathcal{F}}$ and $z = \gamma^{-1} w \in \overline{\mathcal{F}}$ which implies $\mathrm{Im}(\gamma^{-1} w) \geq \mathrm{Im}(w)$. So by (2.1) we have that

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} = \mathrm{Im}(\gamma z) \geq \mathrm{Im}(z) \implies |cz+d| \leq 1. \tag{2.2}$$

Writing $z = x + iy$ and noting the definition of $\mathcal{F}$, we see that $y = \mathrm{Im}(z) \geq \frac{\sqrt{3}}{2}$. Thus, we deduce that

$$\frac{3}{4}c^2 \leq y^2 c^2 \leq (cy)^2 + (cx+d)^2 = |cz+d|^2 \leq 1,$$

which implies $c = -1, 0, 1$. We treat each case separately:

i) For $c = 0$, we have that $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. We see, using (2.2), that $d = -1, 0, 1$. For $d = 0$, the matrix γ is not invertible, which contradicts the fact that $\gamma \in \Gamma$. For $d = \pm 1$, we must have $a = \pm 1$, as $\det(\gamma) = 1$. Hence, $\gamma z = z \pm b$. Since $\gamma z, z \in \overline{\mathcal{F}}$, which has a horizontal width of one, we must have either $b = 0$ in which case $\gamma = \pm I$ and $\gamma z = z$, **or** $b = \pm 1$ and so $\mathrm{Re}(z) = \pm 1/2$ and $\mathrm{Re}(\gamma z) = \mp 1/2$, and so z and γz lie on $\partial \mathcal{F}$.

ii) For $c = 1$, we have that $|z + d| \leq 1$, by (2.2).

   (a) For $d = 0$, the matrix γ is of the form $\gamma = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$. In addition, we have that $|z| = 1$, i.e. z lies on the unit circle. Also, we know that $\det(\gamma) = 1$, which forces $b = -1$. Hence, $\gamma z = a - \frac{1}{z} \in \overline{\mathcal{F}}$. But, since $-1/z$ is an inversion in the unit circle composed with a reflection in the imaginary axis, we must have $-1/z$ lies on the unit circle as well. This means either $a = 0$ or $a = \pm 1$. This gives for $a = 0$ that $\gamma = R$ and both z and γz lie on the unit circle, with γz a reflection of z in the imaginary axis. For $a = 1$, we have that

$z = \rho$ and $\gamma z = \rho^2$ and for $a = -1$, we have that $z = \rho^2$ and $\gamma z = \rho$. Here, $\rho = (1 + \sqrt{-3})/2$, and both ρ and ρ^2 lie in the 'corners' of our domain $\overline{\mathcal{F}}$; see Figure 2.0.

(b) For $d \neq 0$ we must have $d = \pm 1$ as once again we note that $\mathcal{F}$ has unit width and $z \in \overline{\mathcal{F}}$. For $d = 1$ we have that $|z + 1| \leq 1$, which implies that $z + 1$ lies on or inside the unit circle. It is not possible to horizontally translate an element of $\overline{\mathcal{F}}$ by a unit into the interior of the unit disc, so $z + 1$ lies on the unit circle. Since $z \in \overline{\mathcal{F}}$, we must have $z = \rho^2$ and $z + 1 = \rho$. In addition, $\det(\gamma) = a - b = 1$ and so

$$\gamma z = \frac{az + (a-1)}{z+1} = a - \frac{1}{z+1} = a + \rho^2 \in \overline{\mathcal{F}}.$$

In conclusion we must have $a = 0$ and $\gamma z = \rho^2$ **or** $a = 1$ and $gz = \rho$. The case for $d = -1$ is similar.

iii) For $c = -1$, we note, as discussed previously, that $-\gamma$ gives the same action as γ. Hence, we can consider $-\gamma = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, with $-c = 1$. However, we have already verified this case in ii).

In summary, we have shown that if both z and gz are in $\overline{\mathcal{F}}$ and $\gamma \neq \pm I$, then both z and γz lie on the boundary $\partial \mathcal{F}$. Hence, we have that there are no distinct elements in $\mathcal{F}$ that can be mapped to each other by an element of Γ. This completes the proof. $\square$

2.3 Decomposing Elements of $SL_2(\mathbb{Z})$

The proof of Theorem 2.2.2 gives us another interesting result essentially for free.

Corollary 2.3.1. *The group Γ is generated by the matrices*

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

i.e. $\Gamma = \langle R, T \rangle$.

Proof. Let $z \in \mathcal{F}$ and $\gamma \in \Gamma$. Define $H = \langle R, T \rangle$ as before. By the first part of the proof of Theorem 2.2.2, we have there exists a $\delta \in H$ such that $\delta\gamma z \in \overline{\mathcal{F}}$. But z does not lie on the boundary of $\mathcal{F}$. Hence, by the second part of the proof of Theorem 2.2.2, we have that $\delta\gamma = \pm I$, giving $\gamma = \pm\delta^{-1}$. Furthermore, note that

$$R^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -I.$$

Hence, $\gamma = \pm\delta^{-1} = \pm I\delta \in H$, which gives the result. $\square$

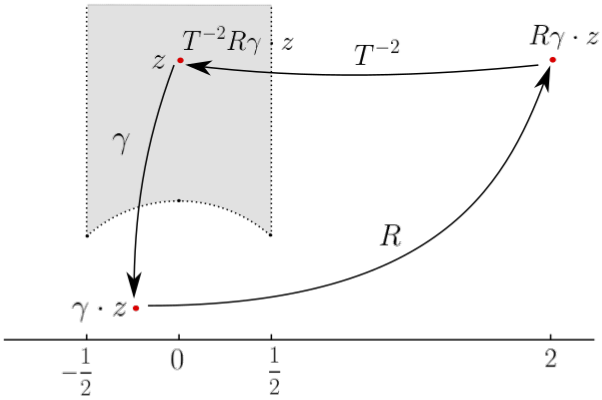

Figure 2.2: Diagram showing the image of $z = 2i$ under the action of $\gamma, R\gamma$ and $T^{-2}R\gamma$, as calculated in Example 2.3.2.

Corollary 2.3.1 states that we can decompose any matrix in Γ into words containing only the letters R, T and their inverses. In fact, Theorem 2.2.2 gives us a specific algorithm for computing such decompositions. Let us first demonstrate this with an example.

Example 2.3.2. Take $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \in \Gamma$ and $z = 2i \in \mathcal{F}$.

$$\gamma z = -1/(2i+2) = (-1+i)/4.$$

So γz lies in $\{w \in \mathbb{H} \mid |\mathrm{Re}(w)| \leq \frac{1}{2}\} \setminus \mathcal{F}$. Applying R to γz gives

$$R\gamma \cdot z = 4/(1-i) = 2+2i.$$

We see that $\mathrm{Im}(R\gamma \cdot z) = 2 > 1$. Hence, we may apply a power of T to translate $R\gamma \cdot z$ into $\mathcal{F}$. Indeed,

$$T^{-2}R\gamma \cdot z = z \in \mathcal{F}.$$

From Theorem 2.2.2, since $z \in \mathcal{F}$, we have that $T^{-2}R\gamma = \pm I$. Calculating $T^{-2}R\gamma$ explicitly, confirms that $T^{-2}R\gamma = -I = R^2$. Using that $R^{-1} = R^3$, we conclude that

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = R^3\, T^2 R^2.$$

Multiplying out the right hand side of this expression does indeed confirm the above conclusion.

We see that the above process is quite easy to follow, but becomes rather laborious to calculate by hand. Therefore, to automate this process, we have written a Python program; see Appendix A.

Example 2.3.3. Let us look at the matrix $\begin{pmatrix} 28 & 9 \\ 59 & 19 \end{pmatrix} \in \Gamma$. It would take quite a while by hand to compute the decomposition of this matrix. However, using the program mentioned above, we can calculate this decomposition by calling

```
wordinRandT(28,9,59,19).
```

This gives us the output

```
[[28,9][59,19]]=R^3T^-2R^3T^9R^3T^-3R^3.
```

Hence, we have that the matrix above decomposes as

$$\begin{pmatrix} 28 & 9 \\ 59 & 19 \end{pmatrix} = R^3\, T^{-2}\, R^3\, T^9\, R^3\, T^{-3}\, R^3.$$

One can indeed check that this is true by multiplying out the right hand side of the above expression.

As a final remark for this chapter, the above shows that the action of Γ on $\mathbb{H}$ is generated by a unit translation parallel to the real axis and an inversion in the unit circle composed with a reflection in the imaginary axis. A reader familiar with hyperbolic geometry will recognise the action of Γ on $\mathbb{H}$ as hyperbolic isometries of the upper half plane model. By the classification of hyperbolic isometries, we know that every isometry can be written as a composition of at most three reflections in hyperbolic lines (reflections in hyperbolic lines in the upper half plane model refer to inversions in circles centred on the real axis and reflections in vertical lines). We see that T and R correspond to a composition of two hyperbolic reflections and are thus orientation preserving. Furthermore, we see that T corresponds to a parabolic isometry (its unique fixed point ∞ lies on the 'boundary' of $\mathbb{H}$) and R corresponds to an elliptic isometry (its unique fixed point i lies in $\mathbb{H}$). One can find more information about hyperbolic geometry in [PT01].

Chapter 3

Modular Forms with Respect to Γ

As alluded to in the previous section, the modular group Γ will play a role in defining the modular forms we will be working with. After some preparatory work, we will define what it means for a function $f : \mathbb{H} \to \mathbb{C}$ to be a modular form with respect to Γ. For this chapter, we will be following [Sch17, Lectures 1–11, pp. 16–17] and [DS05, pp. 1–7].

3.1 Weakly Modular Functions with Respect to Γ

First we will define what it means for a complex-valued function on the upper half plane to be weakly modular with respect to Γ.

Definition 3.1.1 (Weakly Modular Function with Respect to Γ)**.** Let $k \in \mathbb{Z}$. We say a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is weakly modular of weight k with respect to Γ if f satisfies

$$f(\gamma z) = (cz + d)^k f(z), \tag{3.1}$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$.

From this definition, we see that a weakly modular function with respect to Γ has many symmetric properties. For example, using the generators of Γ, the matrices T and R (cf. Corollary 2.3.1), we see that

$$f(Tz) = f(z+1) = f(z) \ \text{ and } \ f(Rz) = f(-1/z) = z^k f(z).$$

In particular, we see that a weakly modular function with respect to Γ is periodic. Moreover, we see that a weakly modular function with respect to Γ of weight zero is invariant under the action of γ, i.e. $f(\gamma z) = f(z)$, for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$.

Remark 3.1.2.

i) In this project, we will be only considering weakly modular functions with respect to Γ. Hence, from now on, for simplicity, we suppress 'with respect to Γ' and refer to weakly modular functions with respect to Γ as simply weakly modular functions.

ii) Note that for $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and f a weakly modular function of weight k, we have that

$$f(\gamma z) = f(z) = (-1)^k f(z).$$

Hence, for k odd we have that $f(z) \equiv 0$. [1]

It turns out we only need to check whether (3.1) is satisfied by T and R, to conclude that a meromorphic function is weakly modular.

Proposition 3.1.3. *For $k \in \mathbb{Z}$, we have that*

$$f \textit{ is weakly modular of weight } k \iff f(z+1) = f(z) \textit{ and } f(-1/z) = z^k f(z).$$

Proof. ($\implies$). Using the elements R and T, the claim follows immediately from the definition of a weakly modular function.
($\Longleftarrow$). We can define a right action $\{f : \mathbb{H} \to \mathbb{C} \mid f \text{ is meromorphic on } \mathbb{H}\} \circlearrowleft \Gamma$ by

$$f(z) \cdot \gamma = (cz+d)^{-k} f(\gamma z), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Indeed, $f(z) \cdot I = 1^{-k} f(Iz) = f(z)$, and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\delta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, with $\gamma, \delta \in \Gamma$, we have that

$$\begin{aligned}(f(z) \cdot \gamma) \cdot \delta &= \Big[(cz+d)^{-k} f(\gamma z)\Big] \cdot \delta = (gz+h)^{-k} \left(c\frac{ez+f}{gz+h} + d\right)^{-k} f(\gamma\delta z) \\ &= [(ce+dg)z + (cf+dh)]^{-k} f(\gamma\delta z) = f(z) \cdot \gamma\delta.\end{aligned}$$

We will see later that this action is a special case of the so called slash operator.

Consider the action of T and R:

$$\begin{aligned} f(z) \cdot T &= f(z+1) = f(z), \\ f(z) \cdot R &= z^{-k} f(-1/z) = f(z). \end{aligned}$$

Hence, using Corollary 2.3.1 and the fact that $\cdot$ defines a right action, we find that

$$f(z) \cdot \gamma = (cz+d)^{-k} f(\gamma z) = f(z), \quad \text{for every } \gamma \in \Gamma.$$

By definition, this shows that $f(z)$ is weakly modular of weight k. □

We conclude this section with a simple but useful fact about weakly modular functions.

[1] Here, some authors, e.g. Serre in [Ser12], define weakly modular functions for only even weights due to this fact.

Lemma 3.1.4. *Let f be weakly modular of weight k and let $\mathcal{F}$ denote the fundamental domain of Γ as defined in Theorem 2.2.2. If $f(z) \neq 0$ for all $z \in \overline{\mathcal{F}}$, then $f(z) \neq 0$ for all $z \in \mathbb{H}$.*

Proof. Suppose $f(w) = 0$, for some $w \in \mathbb{H} \setminus \overline{\mathcal{F}}$. By Theorem 2.2.2, there exists a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $\gamma w \in \overline{\mathcal{F}}$. Using that f is weakly modular of weight k, we see that

$$f(\gamma w) = (cw + d)^k f(w) = 0.$$

This gives a contradiction since $\gamma w \in \overline{\mathcal{F}}$ and we assumed $f(z)$ does not have any zeroes in $\overline{\mathcal{F}}$. $\square$

This illustrates the fact that we can understand properties of a weakly modular function on $\mathbb{H}$ just by knowing how the function behaves on $\overline{\mathcal{F}}$.

3.2 Modular Forms with Respect to Γ

If we require that a weakly modular function has some additional properties, we can define what is called a modular form with respect to Γ.

Suppose we have a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, with $f(z+1) = f(z)$. Consider the holomorphic function

$$q : \mathbb{H} \to \mathbb{D} \setminus \{0\}, \ z \mapsto e^{2\pi i z},$$

where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Indeed we have $q : \mathbb{H} \to \mathbb{D} \setminus \{0\}$ since

$$e^{2\pi i z} = e^{-2\pi \operatorname{Im}(z)} \big[\cos(2\pi \operatorname{Re}(z)) + i \sin(2\pi \operatorname{Re}(z)) \big] \implies \left| e^{2\pi i z} \right| = \left| e^{-2\pi \operatorname{Im}(z)} \right| < 1.$$

Note that $e^{2\pi i z} \neq 0$, for all $z \in \mathbb{C}$. If we have two distinct elements z and w in $\mathbb{H}$ such that $q(z) = q(w)$, then

$$e^{2\pi i z} = e^{2\pi i w} \implies e^{2\pi i (z-w)} = 1 \implies z = w + k, \text{ for some } k \in \mathbb{Z} \setminus \{0\}.$$

However since $f(z+1) = f(z)$, we have $f(z) = f(w)$. Thus, $f(z)$ only depends on the value of q. Hence, there exists a well-defined holomorphic function, $\tilde{f} : \mathbb{D} \setminus \{0\} \to \mathbb{C}$, such that $\tilde{f}(e^{2\pi i z}) = f(z)$ ($\tilde{f}$ is holomorphic since f and q are). Since $\tilde{f}$ is holomorphic on the punctured unit disc (an annulus), it has a Laurent expansion

$$\tilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n.$$

Definition 3.2.1 (Fourier/q-expansion)**.** Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function such that $f(z+1) = f(z)$, for all $z \in \mathbb{H}$ and define $q : \mathbb{H} \to \mathbb{D} \setminus \{0\}, \ z \mapsto e^{2\pi i z}$. We define the Fourier expansion or q-expansion of $f(z)$ as

$$f(z) = \tilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n, \text{ where } q = q(z) = e^{2\pi i z}.$$

Note that $|q(z)| = e^{-2\pi \operatorname{Im}(z)}$. From this we see that $q \to 0$ as $\operatorname{Im}(z) \to \infty$. Intuitively we can view infinity living way off vertically in the imaginary direction in $\mathbb{H}$, with $f(\infty)$ being $\tilde{f}(0)$. This leads us to define what it means for an f as defined above to be holomorphic at infinity.

Definition 3.2.2 (Holomorphic at Infinity)**.** With setting as in Definition 3.2.1, we say such an f is holomorphic at infinity if $\tilde{f}(q)$ extends to a holomorphic function at $q = 0$. Specifically, the q-expansion of f takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

i.e. $a_n = 0$ for all $n < 0$.

Remark 3.2.3.

1. For an f satisfying the conditions of the above definition, we see that $\tilde{f}(q)$ has a removable singularity as $q = 0$. In particular, declaring $\tilde{f}(0) = a_1$, we can extend $\tilde{f}$ to a holomorphic function on $\mathbb{D}$. Moreover, such an extension is unique, using the uniqueness of analytic continuation.

2. Although this seems like quite a technical definition, it is actually quite easy to check in practice. Indeed, for $f(z)$ a holomorphic weakly modular function,

$$f(z) \text{ is holomorphic at infinity} \iff \lim_{\operatorname{Im}(z)\to\infty} |f(z)| \leq R\ , \text{ for some } R \geq 0.$$

   So this is saying that $f(z)$ is holomorphic at infinity if and only if $f(z)$ is bounded as $\operatorname{Im}(z)$ tends to infinity. The proof of each direction follows easily by simply looking at the q-expansion of such an $f(z)$ and taking limits, using the uniform convergence of the Laurent expansion.

With this new definition, we can finally define modular forms with respect to Γ.

Definition 3.2.4 (Modular Form with Respect to Γ)**.** For $f : \mathbb{H} \to \mathbb{C}$, we say f is a modular form of weight $k \in \mathbb{Z}$ with respect to Γ if f satisfies the following conditions:

i) f is holomorphic on $\mathbb{H}$.

ii) f is weakly modular of weight k.

iii) f is holomorphic at infinity.

We let $\mathcal{M}_k(\Gamma)$ denote the set of all modular forms of weight k with respect to Γ.

A special subset of $M_k(\Gamma)$ is the set of cusp forms.

Definition 3.2.5 (Cusp Form)**.** Let $f(z) \in \mathcal{M}_k(\Gamma)$, with q-expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$. We say $f(z)$ is a cusp form if $a_0 = 0$. We let $\mathcal{S}_k(\Gamma)$ denote the space of all cusp forms of weight k.

Remark 3.2.6. One can define different types of modular forms with respect to different discrete subgroup of $SL_2(\mathbb{R})$. A major collection of examples would be modular forms with respect to 'congruence subgroups' of $SL_2(\mathbb{Z})$. For information about modular forms with respect to congruence subgroups, see Chapter 3, Section 3 of [Kob12]. In this project, we will be sticking to modular forms with respect to Γ. For this reason, from now on, when we refer to a modular form we mean a modular form with respect to Γ.

To be a modular form, a function must satisfy some fairly strong conditions. We see that constant functions on $\mathbb{H}$ are weight zero modular forms and that zero function is a modular form of every weight. However, for non-zero weights, it is not immediately clear that such functions even exist. In fact they do, and we will now give a class of important examples, the Eisenstein series.

Definition 3.2.7 (Eisenstein Series)**.** For an integer $k \geq 2$, we define the Eisenstein series of weight k as

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}\ , \quad \text{for } z \in \mathbb{H}.$$

For $k > 2$, we claim that the series above is absolutely convergent. Indeed, one can convert $G_k(z)$ into a sum over the lattice $\Lambda_z = \mathbb{Z}z \oplus \mathbb{Z}$. It thus suffices to show this 'converted' series converges absolutely. For this see [Was08, Lemma 9.4]. Using this fact, for $k > 2$, we may rearrange the series defining $G_k(z)$. Note that as (m, n) runs through $\mathbb{Z}^2 \setminus \{(0, 0)\}$, so does $(-m, -n)$. Using this, we have that

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(-mz-n)^k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{(-1)^k}{(mz+n)^k} = (-1)^k G_k(z).$$

Hence, we conclude for odd $k > 2$ that $G_k(z) \equiv 0$.

So for k odd, G_k is not very interesting. However, for even $k \geq 4$ we have that $G_k \in \mathcal{M}_k(\Gamma)$. So this will give us our first set of non-trivial examples of a modular forms.

Theorem 3.2.8. *Let $k \geq 4$ be an even integer, then $G_k(z) \in \mathcal{M}_k(\Gamma)$ with q-expansion given by*

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{k=1}^{\infty} \sigma_{k-1}(n) q^n\right),$$

where B_k denotes the kth Bernoulli number and $\sigma_{k-1}(n) = \sum_{1\leq d|n} d^{k-1}$ is the weight $(k-1)$ power divisor function.

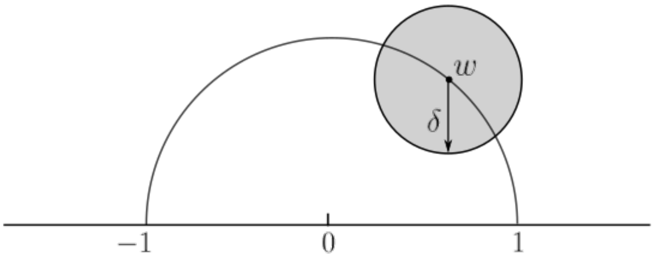

Figure 3.1: The disc, $B_\delta(w)$, of radius δ around w.

Proof. The proof is very standard in the literature, so we omit it. For a full detailed account of the proof, see [Kob12, pp. 109–111]. □

Remark 3.2.9. You may ask, what about the case $k = 2$? One can try to define the Eisenstein series for $k = 2$, however it turns out that G_2 is not a modular form; it fails to be absolutely convergent and as a result, fails the requirement that $G_2(-1/z) = z^2 G_2(z)$. See [Kob12, p. 112] for a more detailed discussion.

To finish this chapter off, let us prove a result for completeness which does not seem to explicitly appear in any of the literature we have seen.

Proposition 3.2.10. *Let $k, l \in \mathbb{Z}$, with $l \neq k$. Then $\mathcal{M}_k(\Gamma) \cap \mathcal{M}_l(\Gamma) = \{0\}$.*

Proof. Let $f \in \mathcal{M}_k(\Gamma) \cap \mathcal{M}_l(\Gamma)$ and assume without loss of generality that $k > l$. Since f is modular of weight k and l, we have that for all $z \in \mathbb{H}$,

$$f(-1/z) = z^k f(z) = z^l f(z) \iff f(z)(z^{k-l} - 1) = 0.$$

We have that z satisfies $z^{k-l} - 1 = 0$ if and only if z is a $(k-l)$th root of unity. In this case we have that $|z| = 1$. Since the above equation must hold any $z \in \mathbb{H}$, we must have that $f(z) = 0$, for all $z \in \mathbb{H}$ with $|z| \neq 1$. Suppose that there is a point $w \in \mathbb{H}$ with $|w| = 1$ and $f(w) \neq 0$. Since f is holomorphic it is also continuous on $\mathbb{H}$. Let $\epsilon = |f(w)| > 0$. Then, by continuity, there exists a $\delta > 0$ such that for all $z \in \mathbb{H}$ satisfying $|z - w| < \delta$,

$$|f(w)| > |f(w) - f(z)| \geq ||f(w)| - |f(z)|| \geq |f(w)| - |f(z)|\,.$$

This implies that $|f(z)| > 0$, for all $z \in B_\delta(w) = \{z \in \mathbb{H} \mid |z - w| < \delta\}$

However, this disc will certainly contain a point v with $|v| \neq 1$. From the above discussion, for such a v, we have that $f(v) = 0$. This gives a contradiction to the fact that $|f(z)| > 0$, for all $z \in B_\delta(w)$, and so we conclude that $f(z) \equiv 0$ on $\mathbb{H}$. □

Chapter 4

The Algebraic Structure of Modular Forms

In this chapter, we will show that we can add more structure to the set $\mathcal{M}_k(\Gamma)$. Namely, we will show that it forms a $\mathbb{C}$-vector space which turns out to be, surprisingly, finite-dimensional. Moreover, we will be able to explicitly derive both its dimension and structure. Finally, we will show how this extra structure helps us determine some interesting results concerning both Ramanujan's Δ-function and the power divisor function. Again the material from this chapter is standard and can be found in any book on the subject of modular forms for $SL_2(\mathbb{Z})$. We will be following [Jer06] and [Sch17, Lectures 1–11, pp. 21–24].

4.1 The Space of Modular Forms

In this section, we will add the aforementioned structure to the sets $\mathcal{M}_k(\Gamma)$ and then discuss the properties of these spaces.

4.1.1 The Vector Space of Modular Forms

To start with, let us show that $\mathcal{M}_k(\Gamma)$ forms a $\mathbb{C}$-vector space.

Proposition 4.1.1. *For $k \in \mathbb{Z}$ we have $\mathcal{M}_k(\Gamma)$ forms a vector space over $\mathbb{C}$.*

Proof. Note that the set of all holomorphic functions on $\mathbb{H}$ forms a vector space over $\mathbb{C}$ and we have that $\mathcal{M}_k(\Gamma)$ is contained in this set.

We have that the zero function is in $\mathcal{M}_k(\Gamma)$, so $\mathcal{M}_k(\Gamma)$ is non-empty. In addition, we claim that for $\lambda, \mu \in \mathbb{C}$ and $f(z), g(z) \in \mathcal{M}_k(\Gamma)$, we have that $(\lambda f + \mu h)(z) \in \mathcal{M}_k(\Gamma)$. Indeed $(\lambda f + \mu g)(z)$ is certainly holomorphic on $\mathbb{H}$. Next, we claim that $(\lambda f + \mu h)(z)$ is also weakly modular of weight k. Firstly, we have that

$$(\lambda f + \mu h)(z + 1) = \lambda f(z + 1) + \mu h(z + 1) = \lambda f(z) + \mu h(z) = (\lambda f + \mu h)(z),$$

$$(\lambda f + \mu h)\left(-\frac{1}{z}\right) = \lambda f\left(-\frac{1}{z}\right) + \mu h\left(-\frac{1}{z}\right) = z^k \lambda f(z) + z^k \mu h(z) = z^k(\lambda f + \mu h)(z).$$

The claim then follows from Proposition 3.1.3. Finally, $(\lambda f + \mu h)(z)$ is holomorphic at infinity since

$$(f + h)(z) = f(z) + h(z) = \lambda \sum_{n=0}^{\infty} a_n e^{2\pi i n z} + \mu \sum_{n=0}^{\infty} b_n e^{2\pi i n z} = \sum_{n=0}^{\infty} (\lambda a_n + \mu b_n) e^{2\pi i n z}.$$

Hence, $(\lambda f + \mu h)(z) \in \mathcal{M}_k(\Gamma)$. This shows that $\mathcal{M}_k(\Gamma)$ is a subspace of the space of holomorphic functions on $\mathbb{H}$, which in turn shows $\mathcal{M}_k(\Gamma)$ is a vector space over $\mathbb{C}$. $\square$

Next, we note an interesting property concerning the product of two modular forms.

Proposition 4.1.2. *Let $k, l \in \mathbb{Z}$. We have for $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_l(\Gamma)$ that $fg \in \mathcal{M}_{k+l}(\Gamma)$.*

Proof. Firstly, $(fg)(z)$ is certainly holomorphic on $\mathbb{H}$ since $f(z)$ and $g(z)$ are. Now, given $f(z) = \sum_{n=0}^{\infty} a_n q^n$ and $g(z) = \sum_{n=0}^{\infty} b_n q^n$, we have that

$$(fg)(z) = f(z)g(z) = \left(\sum_{n=0}^{\infty} a_n q^n\right)\left(\sum_{n=0}^{\infty} b_n q^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right) q^n, \tag{4.1}$$

which shows $(fg)(z)$ extends to a holomorphic function at $q = 0$. Finally, we see that

i) $(fg)(z+1) = f(z+1)g(z+1) = f(z)g(z) = (fg)(z)$,

ii) $(fg)(-1/z) = f(-1/z)g(-1/z) = z^k f(z) \cdot z^l g(z) = z^{k+l}(fg)(z)$.

Using Proposition 3.1.3, this shows that $(fg)(z)$ is weakly modular of weight $(k+l)$ and hence we conclude that $(fg)(z) \in M_{k+l}(\Gamma)$. $\square$

Using Proposition 3.2.10 and Proposition 4.1.2, we see that

$$\mathcal{M}_*(\Gamma) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(\Gamma)$$

forms a so called graded ring, with additive identity the zero function and multiplicative identity the constant function $1(z) = 1$, with $1(z) \in \mathcal{M}_0(\Gamma)$ (graded refers to the fact that the direct sum is indexed over the integers). Furthermore, define

$$\mathcal{S}_*(\Gamma) = \bigoplus_{k \in \mathbb{Z}} \mathcal{S}_k(\Gamma)$$

as the graded ideal of cusp forms. By definition, $\mathcal{S}_*(\Gamma) \subset \mathcal{M}_*(\Gamma)$. Moreover, $\mathcal{S}_*(\Gamma)$ is an ideal in $\mathcal{M}_*(\Gamma)$ since for $\phi : \mathcal{M}_*(\Gamma) \to \mathbb{C}$, where $f(z) = \sum_{n \geq 1} a_n q^n \mapsto a_0$, we have that

$$\mathcal{S}_*(\Gamma) = ker(\phi).$$

We have that ϕ is a ring homomorphism and the kernel of a ring homomorphism is an ideal in the domain of the homomorphism. Note that ϕ is also surjective (consider any constant function in $\mathcal{M}_0(\Gamma)$). So, by the first isomorphism theorem, we have as rings that

$$\mathcal{M}_*(\Gamma)/\mathcal{S}_*(\Gamma) \cong \mathbb{C}.$$

Remark 4.1.3. We can say a little more about multiplying modular forms together. Indeed if $f(z) \in \mathcal{S}_k(\Gamma)$ and $g(z) \in \mathcal{M}_l(\Gamma)$, then $(fg)(z) \in \mathcal{S}_{k+l}(\Gamma)$. This follows from (4.1).

4.1.2 Valence Formula

As we now know that $\mathcal{M}_k(\Gamma)$ forms a vector space, it is natural to wonder whether such a space is finite-dimensional. We have already seen that $\dim \mathcal{M}_k(\Gamma) = 0$, for any odd integer k. Interestingly, we will see that $\mathcal{M}_k(\Gamma)$ is indeed finite-dimensional for any k. However, it will take some work to arrive at this result.

Recall from complex analysis that if $f : U \to \mathbb{C}$ is holomorphic on U, an open set, then for any $a \in U$ and $r > 0$ such that $B_r(a) \subseteq U$, we have that $f(z)$ has a convergent Taylor series representation around $z = a$, i.e. $f(z) = \sum_{n\geq 0} a_n(z-a)^n$, for $z \in B_r(a)$. If $z = a \in U$ is a zero of $f(z)$, then the order of $f(z)$ at $z = a$, denoted $\mathrm{ord}_a(f)$, is the smallest $n \in \mathbb{N}$ such that $a_n \neq 0$. Next, we define the order of the zero of f at infinity as $\mathrm{ord}_\infty(f) = \mathrm{ord}_0(\tilde{f})$, i.e. the smallest $n \in \mathbb{N}$ such that the coefficient of the nth term in the q-expansion of f is non-zero.

Example 4.1.4. The Eisenstein series E_k has a constant term of $2\zeta(k)$ in its q-expansion and hence does not have a zero a infinity. Later, we will see an important example of a cusp form, namely Ramanujan's Δ-function, which has a q-expansion given by $\Delta(z) = \sum_{n\geq 1} \tau(n)q^n$. For Ramanujan's Δ-function, we have that $\mathrm{ord}_\infty(\Delta) = 1$, since it turns out that $\tau(1) = 1$; see Definition 4.1.14.

It will be important to know how the action of $\gamma \in \Gamma$ affects the order of a zero of $f \in \mathcal{M}_k(\Gamma)$.

Lemma 4.1.5. *For $f \in \mathcal{M}_k(\Gamma)$ and $\gamma \in \Gamma$, we have $\mathrm{ord}_{\gamma a}(f) = ord_a(f)$.*

Proof. Using that f is is weakly modular we have that $f(\gamma z) = (cz+d)^k f(z)$, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Let $m = \mathrm{ord}_a(f)$, in particular $f(z)$ has Taylor series around $z = a$ of the form $f(z) = \sum_{n\geq m} a_n z^n$, with $a_m \neq 0$. Note that $cz + d \neq 0$ for $z \in \mathbb{H}$, as $-d/c \notin \mathbb{H}$, for $d, c \in \mathbb{Z}$. Hence, we have that

$$\begin{aligned} f(\gamma z) = (cz+d)^k f(z) &= \Big[(ca+d)^k + kc(ca+d)^{k-1}(z-a) + \cdots\Big] \cdot \Big[a_m(z-a)^m + \cdots\Big] = \\ &= a_m(ca+d)^k(z-a)^m + \cdots, \end{aligned}$$

where we have used the Taylor series of $(cz+d)^k$ about $z = a$. As a_m and $(ca+d)$ are not zero, this completes the proof. □

Lemma 4.1.6. *For $0 \neq f \in \mathcal{M}_k(\Gamma)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have that*

$$\int_{\gamma C} \frac{f'(z)}{f(z)} dz - \int_C \frac{f'(z)}{f(z)} dz = kc \int_C \frac{1}{cz+d} dz.$$

Proof. Since $f \in \mathcal{M}_k(\Gamma)$, we have $f(\gamma z) = (cz+d)^k f(z)$. Differentiating both sides of this expression with respect to z, we get that

$$\begin{aligned} f'(\gamma z)\frac{d(\gamma z)}{dz} &= kc(cz+d)^{k-1}f(z) + (cz+d)^k f'(z) \\ \implies \frac{f'(\gamma z)}{f(\gamma z)}\frac{d(\gamma z)}{dz} &= kc\frac{(cz+d)^{k-1}}{f(z)(cz+d)^k} + \frac{f'(z)(cz+d)^k}{f(z)(cz+d)^k} = \frac{f'(z)}{f(z)} + \frac{ck}{cz+d}. \end{aligned}$$

Now, working with the the left hand side of the claim, we see that

$$\begin{aligned} \int_{\gamma C} \frac{f'(z)}{f(z)} dz - \int_C \frac{f'(z)}{f(z)} &= \int_C \frac{f'(\gamma z)}{f(\gamma z)} d(\gamma z) - \int_C \frac{f'(z)}{f(z)} dz \\ &= \int_C \frac{f'(\gamma z)}{f(\gamma z)} \frac{d(\gamma z)}{dz} dz - \int_C \frac{f'(z)}{f(z)} dz = kc \int_C \frac{1}{cz+d} dz. \end{aligned}$$

□

The next result will play a role crucial in understanding the structure of $\mathcal{M}_k(\Gamma)$.

Theorem 4.1.7. *(Valence Formula). If $f \in \mathcal{M}_k(\Gamma)$ is not identically zero, then we have that*

$$\sum_{z \in \overline{\mathcal{F}} \setminus \{i, \rho\, \rho^2\}} \mathrm{ord}_z(f) + \mathrm{ord}_\infty(f) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) = \frac{k}{12}\,.$$

.

Proof. The idea for the proof is to integrate around $\partial\mathcal{F}$ avoiding $i,\ \rho,\ \rho^2$ and any zeroes of f by going around these points with small circular arcs of radius ϵ then taking the limit as $\epsilon \to 0$. Note that by having one zero, $z \in \partial\mathcal{F} \setminus \{i, \rho, \rho^2\}$, we will automatically have another symmetrically placed zero in $\partial\mathcal{F} \setminus \{i, \rho, \rho^2\}$ coming from the action of some $\gamma \in \Gamma$ (cf. the proof of Theorem 2.2.2). We would like to avoid this point as well, so we use the image of the circular arc around z under the action of γ to do so. Note this means that we only ever count a zero of f in $\overline{\mathcal{F}} \setminus \{i, \rho, \rho^2\}$ once, as in the limit only one of γz or z is enclosed in the interior of our contour.

Since f is holomorphic at infinity, we can find an $R > 0$ such that f has no zeroes in $\{z \in \mathbb{H} \mid \mathrm{Im}(z) \geq R\}$. This follows from the principle of isolated zeroes which tells us that there exists an $r > 0$, such that $\tilde{f}(q) \neq 0$ in $\{q \in \mathbb{D} \mid |q| < r \leq 1\}$. So choose an R with $e^{-2\pi R} < r$. For such an R, we have that $\tilde{f}(q) \neq 0$ in $\{q \in \mathbb{D} \mid |q| \leq e^{-2\pi R}\}$ and so $f(z)$ is not equal to zero in $\{z \in \mathbb{H} \mid \mathrm{Im}(z) \geq R\}$.

Let $\mathcal{F}_R = \{z \in \mathcal{F} \mid \operatorname{Im}(z) \leq R\}$ and denote the set of zeroes in $\overline{\mathcal{F}_R}$ by $Z(f)$. Let us show that the claim is well-defined. To see this, note that $\overline{\mathcal{F}_R}$ is a compact subset of $\mathbb{C}$. If f had an infinite number of zeroes in $\overline{\mathcal{F}_R}$, then by Bolzano-Weierstraß, $Z(f)$ has a limit point in $\overline{\mathcal{F}_R}$. By the identity theorem, we thus have that $f(z) \equiv 0$. This contradicts our assumption that f is not identically zero. So, we conclude that f only has a finite number of zeroes in $\overline{\mathcal{F}_R}$.

The rest of the proof is an exercise in contour integration. Details can be found in Appendix B. $\square$

4.1.3 The Dimension and Structure of the Space of Modular Forms

Using the valence formula, we can prove that the dimension of the complex vector space $\mathcal{M}_k(\Gamma)$ is finite. First though, let us prove some some immediate results, demonstrating how to apply the valence formula.

Corollary 4.1.8. *If $k < 0$, then $\mathcal{M}_k(\Gamma) = 0$. In particular* $\dim \mathcal{M}_k(\Gamma) = 0$.

Proof. For $0 \neq f \in \mathcal{M}_k(\Gamma)$ with $k < 0$, the right hand side of the valence formula is less than zero whereas, since f has no poles in $\mathbb{H}$ as it is holomorphic on $\mathbb{H}$, the left hand side of the valence formula is at least zero. This gives a contradiction, meaning we must have that f is identically zero. $\square$

Corollary 4.1.9. *We have that $\mathcal{M}_0(\Gamma) = \mathbb{C}$. In particular,* $\dim \mathcal{M}_0(\Gamma) = 1$.

Proof. Let $0 \neq f \in \mathcal{M}_0(\Gamma)$ and c be a value taken by f in $\overline{\mathcal{F}}$. Define the function $g(z) = f(z) - c$, which is also an element of $\mathcal{M}_0(\Gamma)$. Using the valence formula for g, we see that the right hand side is zero, whereas the left hand side is greater than zero, since g has a zero in $\overline{\mathcal{F}}$ by construction. Again we arrive at a contradiction, giving that $g(z) = f(z) - c \equiv 0$, and thus $f(z) = c$, for all $z \in \mathbb{H}$. Since any constant function is an element of $\mathcal{M}_0(\Gamma)$, we have that $\mathcal{M}_0(\Gamma) = \mathbb{C}$. $\square$

The above results show just how useful the valence formula is. We will now use it again to show that the dimension of the space $\mathcal{M}_k(\Gamma)$ is finite for all $k \in \mathbb{Z}$.

Theorem 4.1.10. $\dim \mathcal{M}_k(\Gamma) \leq \left\lfloor \frac{k}{12} \right\rfloor + 1$*, for every integer k.*

Proof. Let $n = \left\lfloor \frac{k}{12} \right\rfloor + 1$ and $f_0, f_1, \cdots, f_n \in \mathcal{M}_k(\Gamma)$. Pick n distinct points, $z_1, \cdots, z_n \in \overline{\mathcal{F}} \setminus \{i, \rho, \rho^2\}$. Then we can find $\lambda_i \in \mathbb{C}$, not all zero, such that the function

$$f(z) := \sum_{i=0}^{n} \lambda_i f_i(z) \in \mathcal{M}_k(\Gamma)$$

vanishes at $z_1, \ldots, z_n$. This follows from the fact that we have a system of n equations in $n+1$ variables, λ_i, and so the rank-nullity theorem tells us that we will have a non-trivial kernel, since the rank of this system of equations is at most n.

Now, using the Valence formula for f, we see that the left hand side is greater than or equal to n, whereas the right hand side is strictly less than n, since $\frac{k}{12} < 1 + \left\lfloor \frac{k}{12} \right\rfloor = n$. This gives a contradiction unless we have that f is identically zero on $\mathbb{H}$. This means the set of $n+1$ elements $\{f_0, \cdots, f_n\}$ are linearly dependent, giving that $\dim \mathcal{M}_k(\Gamma) \leq n = \left\lfloor \frac{k}{12} \right\rfloor + 1$. □

Remark 4.1.11.

i) Theorem 4.1.10 tells us that $\mathcal{M}_k(\Gamma)$ is finite-dimensional for any integer k. In addition, Theorem 4.1.10 gives a bound on the dimension of the space $\mathcal{M}_k(\Gamma)$ that grows relatively slowly as k increases. For example, we see for $k \in \{1, 2, \cdots, 11\}$ that $\dim \mathcal{M}_k(\Gamma) = 0$ or 1. This 'confinement' of modular forms leads to many interesting results and relations, some of which we will see later on in this chapter.

ii) Although at first demanding that $f \in \mathcal{M}_k(\Gamma)$ be holomorphic at infinity may have seemed like technical nonsense, it was very important in proving the valence formula and thus plays a major role in why $\mathcal{M}_k(\Gamma)$ is finite-dimensional.

In the previous section we introduced the Eisenstein series and have seen its q-expansion. Now we would like to normalise this series.

Definition 4.1.12 (Normalised Eisenstein Series)**.** For an even integer $k \geq 4$ we define the Normalised Eisenstein Series of weight k as

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)}\ , \quad \text{for } z \in \mathbb{H},$$

with q-expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Q}[[q]].$$

We have already seen that $G_k \in \mathcal{M}_k(\Gamma)$. Since we also know that $\mathcal{M}_k(\Gamma)$ forms a $\mathbb{C}$-vector space, we have that $E_k \in \mathcal{M}_k(\Gamma)$. By Theorem 4.1.10, we know that $\dim \mathcal{M}_k(\Gamma) = 0$ or 1, for $k = 4, 6, 8, 10$. Since $0 \neq E_k(z) \in \mathcal{M}_k(\Gamma)$, we see that $\dim \mathcal{M}_k(\Gamma) = 1$ with

$$\mathcal{M}_k(\Gamma) = \langle E_k \rangle = \mathbb{C}E_k\ , \ \text{ for } k = 4, 6, 8, 10.$$

Unfortunately, since $E_2 \notin \mathcal{M}_2(\Gamma)$ (cf. Remark 3.2.9), we cannot use the above reasoning to establish the structure of $\mathcal{M}_2(\Gamma)$. However, appealing directly to the valence formula, we can determine this case as well.

Lemma 4.1.13. *The space $\mathcal{M}_2(\Gamma)$ is trivial, i.e. $\mathcal{M}_2(\Gamma) = 0$.*

Proof. Suppose, for a contradiction we have an $f \in \mathcal{M}_2(\Gamma)$ that is not the zero function. Let $a = \sum_{z \in \overline{\mathcal{F}} \setminus \{i, \rho\, \rho^2\}} \operatorname{ord}_z(f) + \operatorname{ord}_\infty(f)$, $b = \operatorname{ord}_i(f)$ and $c = \operatorname{ord}_\rho(f)$. Note that these are all non-negative integers. Applying the valence formula to f, we see that

$$\frac{1}{6} = a + \frac{b}{2} + \frac{c}{3} \iff 1 = 6a + 3b + 2c.$$

But this equation has no non-negative solutions, which gives the desired contradiction. $\square$

Although we have defined cusp forms, we are yet to see an example of one. By Proposition 4.1.2, we have that $E_4^3, E_6^2 \in \mathcal{M}_{12}(\Gamma)$. Since $\mathcal{M}_{12}(\Gamma)$ is a vector space, $E_4^3 - E_6^2 \in \mathcal{M}_{12}(\Gamma)$. Now consider the q-expansion of this expression:

$$(E_4^3 - E_6^2)(z) = (1 + 240q + \cdots)^3 - (1 - 504q + \cdots)^2 = 1728q - 41472q^2 + 435456q^3 + \cdots .$$

Hence, we have that $E_4^3 - E_6^2 \in \mathcal{S}_{12}(\Gamma)$, with $E_4^3 - E_6^2$ not the zero function. Let us normalise this and define the following cusp form:

Definition 4.1.14 (Ramanujan's Δ-Function)**.** We define Ramanujan's Δ-function as $\Delta = (E_4^3 - E_6^2)/1728 \in \mathcal{S}_{12}(\Gamma)$, with q-expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - \cdots \in \mathcal{S}_{12}(\Gamma).$$

The function $\tau : \mathbb{N} \to \mathbb{Q}$ is known as Ramanujan's τ-function.

Note the following important property of the Δ-function.

Lemma 4.1.15. *$\Delta(z) \neq 0$, for all $z \in \mathbb{H}$.*

Proof. By construction, we have that $\operatorname{ord}_\infty(\Delta) = 1$. In addition, as noted above, $\Delta(z) \not\equiv 0$. Now, looking at the valence formula applied to Δ, we see that the right hand side is equal to one and so $\Delta(z)$ cannot have any zeroes in $\overline{\mathcal{F}}$. By Lemma 3.1.4, this is sufficient to show that $\Delta(z) \neq 0$ for any $z \in \mathbb{H}$. $\square$

We know a lot about modular forms of weight less than twelve. The idea now is to use the information we have on these low weight spaces to gain information about higher weight spaces. The following is the main result we will need to do this.

Lemma 4.1.16. *Let $k \geq 12$ be an even integer. Then we have $\mathcal{S}_k(\Gamma) \cong \mathcal{M}_{k-12}(\Gamma)$ as $\mathbb{C}$-vector spaces.*

Proof. Define the map

$$\phi : \mathcal{M}_{k-12} \to \mathcal{S}_k(\Gamma), \; f \mapsto \Delta f.$$

We can immediately see that ϕ is a linear map. Moreover, note that ϕ is well-defined (i.e. maps into $\mathcal{S}_k(\Gamma)$), which follows from Proposition 4.1.2 and Remark 4.1.3.

By Lemma 4.1.15, if $\Delta f = 0$, then $f = 0$. As a result, the kernel of ϕ is trivial, and so ϕ is injective.

Let $g \in \mathcal{S}_k(\Gamma)$ then we have $\phi(g/\Delta) = g$. Is g/Δ and element of $\mathcal{M}_{k-12}(\Gamma)$? Well, using Lemma 4.1.15 , we see that $(g/\Delta)(z)$ is holomorphic in $\mathbb{H}$. In addition, we have that $(g/\Delta)(z+1) = (g/\Delta)(z)$ and $(g/\Delta)(-1/z) = z^{k-12}(g/\Delta)(z)$, and so g/Δ is weakly modular of weight $(k-12)$. Finally, looking at the q-expansion of g/Δ, we see that

$$\left(\frac{g}{\Delta}\right) = \frac{a_1 q + a_2 q^2 + \cdots}{q - 24q^2 + \cdots} \to a_1, \text{ as } q \to 0.$$

Hence g/Δ is holomorphic at infinity. Thus, we conclude that $g/\Delta \in \mathcal{M}_{k-12}(\Gamma)$ and so ϕ is surjective, and thus an isomorphism. $\square$

Using this lemma, we can deduce the general structure of the space of modular forms.

Proposition 4.1.17. *For $k \geq 4$, an even integer, we have that*

$$\mathcal{M}_k(\Gamma) = \mathcal{S}_k(\Gamma) \oplus \mathbb{C}E_k.$$

Proof. Since $E_k \in \mathcal{M}_k(\Gamma)$ for even $k \geq 4$, we have a subspace $\langle E_k \rangle = \mathbb{C}E_k \leq \mathcal{M}_k(\Gamma)$. Now consider the projection

$$\pi : \mathcal{M}_k(\Gamma) \to \mathbb{C}E_k, \ f \mapsto a_0 E_k,$$

where $f(z) = \sum_{k=0}^{\infty} a_k q^k$ is the q-expansion of f. Indeed this is a projection since this map is linear and

$$\pi(\lambda E_k(z)) = \pi\left(\lambda - \lambda \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n\right) = \lambda E_k(z).$$

We see that $\ker(\pi) = \mathcal{S}_k(\Gamma)$. Hence, since π is a projection, we have using a well know result from linear algebra that

$$\mathcal{M}_k(\Gamma) = \ker(\pi) \oplus \mathbb{C}E_k = \mathcal{S}_k(\Gamma) \oplus \mathbb{C}E_k.$$

$\square$

Now, using this information on the structure of the space $\mathcal{M}_k(\Gamma)$, we can finally deduce its dimension for all $k \in \mathbb{Z}$.

Theorem 4.1.18 (Dimension Formula)**.** *We have that*

$$\dim \mathcal{M}_k(\Gamma) = \begin{cases} 0 & \text{for } k \text{ odd or } k < 0, \\ \lfloor \frac{k}{12} \rfloor & \text{for } k > 0 \text{ and } k \equiv 2 \ (\text{mod}\, 12), \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Proof. From Corollary 4.1.8, the claim holds for $k < 0$, so assume $k \geq 0$. Let $k = 12n + r$, for $0 \leq r \leq 11$. In addition, we know that for k odd, the claim holds (cf. Remark 3.1.2), so we restrict to r being even. For $n = 0$ the claim agrees with our previous work, which provides the base case.

Assume the claim is true for some $n \geq 0$. Applying Lemma 4.1.16, we observe that

$$\dim \mathcal{S}_{12(n+1)+r}(\Gamma) = \dim \mathcal{M}_{12n+r}(\Gamma) = \begin{cases} \lfloor \frac{12n+r}{12} \rfloor & \text{for } r = 2, \\ \lfloor \frac{12n+r}{12} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Hence using Proposition 4.1.17, we see that

$$\dim \mathcal{M}_{12(n+1)+r}(\Gamma) = \dim \mathcal{S}_{12n+r}(\Gamma) + 1 = \begin{cases} \lfloor \frac{12n+r}{12} \rfloor + 1 & \text{for } r = 2, \\ \lfloor \frac{12n+r}{12} \rfloor + 2 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \lfloor \frac{12(k+1)+r}{12} \rfloor & \text{for } r = 2, \\ \lfloor \frac{12(k+1)+r}{12} \rfloor + 1 & \text{otherwise.} \end{cases}$$

So the result follows by induction on n. $\square$

4.1.4 A Basis for the Space of Modular Forms

The dimension formula is an extremely useful result. We will see some immediate consequences of this result in the next section, but first we would like to find an explicit basis for $\mathcal{M}_k(\Gamma)$.

Proposition 4.1.19. *For $k \in \mathbb{N}_0$, the space $\mathcal{M}_k(\Gamma)$ has a basis*

$$\begin{cases} \{\Delta^i E_4^{(k-12i)/4} \mid 0 \leq i < \dim \mathcal{M}_k(\Gamma)\} & \text{for } k \equiv 0 \ (mod\, 4), \\ \{\Delta^i E_4^{(k-12i-6)/4} E_6 \mid 0 \leq i < \dim \mathcal{M}_k(\Gamma)\} & \text{for } k \equiv 2 \ (mod\, 4). \end{cases}$$

Proof. Hopefully this seems plausible to the reader. A full proof can be found in Appendix C. $\square$

Example 4.1.20. Consider the space $\mathcal{M}_{36}(\Gamma)$. First, note that $36 \equiv 0 \pmod{12}$. So, the dimension formula tell us that $\dim \mathcal{M}_{36}(\Gamma) = \lfloor \frac{36}{12} \rfloor + 1 = 4$. So, $\mathcal{M}_k(\Gamma)$ is a four dimensional space. Using Proposition 4.1.19, we have that

$$\{E_4^9, \Delta E_4^6, \Delta^2 E_4^3, \Delta^3\}$$

is a basis of $\mathcal{M}_{36}(\Gamma)$. Moreover, we know from Proposition 4.1.17 that $\dim \mathcal{S}_k(\Gamma) = \dim \mathcal{M}_k(\Gamma) - 1 = 3$. Note, E_4^9 is not a cusp form since it has constant term of one in its q-expansion. Hence, we conclude that

$$\{\Delta E_4^6, \Delta^2 E_4^3, \Delta^3\}$$

is a basis for $\mathcal{S}_{36}(\Gamma)$, since a set of $\dim \mathcal{S}_{36}(\Gamma)$ linearly independent vectors must form a basis of $\dim \mathcal{S}_{36}(\Gamma)$.

4.2 Applications of the Dimension Formula

In this section we will illustrate how one one can use the dimension formula to derive an interesting results concerning Ramanujan's Δ-function and another result which relates certain power divisor functions.Throughout this section, we will use the following definition:

$$S_k(q) := \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \text{i.e. } E_k(q) = 1 - \frac{2k}{B_k} S_k(q).$$

4.2.1 Properties of Ramanujan's τ-Function

Recall in the last section we defined Ramanujan's Δ-function as

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = \sum_{n=1}^{\infty} \tau(n) q^n \in \mathcal{S}_{12}(\Gamma),$$

where the coefficients of the q-expansion of $\Delta(z)$ are given by $\tau : \mathbb{N} \to \mathbb{Q}$, Ramanujan's$\tau$-function. A priori there doesn't seem anything special about $\tau(n)$. However, using our previous work, we can derive some very interesting properties of this function.

Proposition 4.2.1. *$\tau(n) \in \mathbb{Z}$, for all $n \in \mathbb{N}$.*

Proof. Using the q-expansions of E_4 and E_6, we have that

$$1728\Delta = (1 + 240\ S_4(q))^3 - (1 - 504\ S_6(q))^2 \in \mathbb{Z}[[q]].$$

We want to show that the $1728 = 12^3$ divides the right hand side of this expression. Expanding the right hand side gives

$$3 \cdot 240\ S_4(q) + 3 \cdot 240^2\ S_4(q)^2 + 240^3\ S_4(q)^3 + 2 \cdot 504\ S_6(q) - 504^2\ S_6(q)^2 \in \mathbb{Z}[[q]].$$

Reducing this expression mod 1728, we obtain

$$720\ S_4(q) + 1008\ S_6(q) \in (\mathbb{Z}/1728\mathbb{Z})[[q]].$$

Note $12^2 \mid 720, 1008$, so it suffices to show that 12 divides the coefficients of $5\ S_4(q) + 7\ S_6(q)$, i.e. we need that $5\sigma_5(n) + 7\sigma_5(n) \equiv 0 \pmod{12}$, for every $n \in \mathbb{N}$. By looking at this expression term by term, we only need to check $5a^3 + 7a^5 \equiv 0 \pmod{12}$ for each positive divisor a of n. Here, we are working modulo twelve, so we only have to check this holds for $a \in \{0, ..., 11\}$. A simple calculation check shows this is indeed satisfied for any $a \in \{0, ..., 11\}$, which gives the result. $\square$

So we have that $\tau(n)$ is an integer-valued function, i.e. $\tau : \mathbb{N} \to \mathbb{Z}$. Famously, Ramanujan observed the surprising congruence relation $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, for all $n \in \mathbb{N}$. Using the dimension formula for modular forms, this result drops out nicely.

Theorem 4.2.2. *$\tau(n) \equiv \sigma_{11}(n)$ $(mod\, 691)$, for all $n \in \mathbb{N}$.*

Proof. We have that , $E_4^3, E_{12} \in \mathcal{M}_{12}(\Gamma)$ with

$$E_4(z)^3 = 1 + \sum_{n=1}^{\infty} a_n q^n \ , \ E_{12}(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.$$

Since $\mathcal{M}_{12}(\Gamma)$ forms a vector space, $E_{12} - E_4^3 \in \mathcal{S}_{12}(\Gamma)$. By the dimension formula, we have that

$$E_{12} - E_4^3 = \lambda\Delta \ , \text{ for some } \lambda \in \mathbb{C}^{\times}.$$

Hence, by comparing coefficients and multiplying by 691, we obtain

$$\mu\tau(n) = 65520\sigma_{11}(n) - 691a_n, \tag{4.2}$$

where $\mu = 691 \cdot \lambda$. We have $\tau(1) = \sigma_{11}(1) = 1$. Hence, $\mu = 65520 - 691a_n \in \mathbb{Z}$ and thus $\mu \equiv 65520 \pmod{691}$. Now reducing (4.2) modulo 691, we see that

$$65520\tau(n) \equiv 65520\sigma_{11}(n) \pmod{691}.$$

Using the Euclidean algorithm we see that the $\gcd(691, 65520) = 1$. Hence, we can cancel 65520 from both sides of the above equation and obtain the claimed result. □

Ramanujan's τ-function satisfies many more congruence relations in a style similar to that of the above congruence. A nice collection of such congruences can be found at the beginning of [SD73].

4.2.2 A Property of the Power Divisor Function, $\sigma_k(n)$

To finish this chapter, we will give an example of a surprising relation between power divisor functions that we can derive by using the dimension formula, Theorem 4.1.18. Specifically, this example will give a solution to a part of problem 3 from Chapter 3, Section 2 of [Kob12].

Example 4.2.3. By Proposition 4.1.2, we have $E_4E_6 \in \mathcal{M}_{10}(\Gamma)$. Moreover, using the dimension formula, we see that $E_4E_6 = \lambda E_{10}$, for some $\lambda \in \mathbb{C}^{\times}$. Writing this expression in terms of series gives

$$\begin{aligned} &(1 + 240S_4(q))(1 - 504S_6(q)) = \lambda(1 - 264S_{10}(q)) \\ \iff &1 - 504S_6(q) + 240S_4(q) - 120960S_4(q)S_6(q) = \lambda(1 - 264S_{10}(q)). \end{aligned}$$

Comparing the constant coefficient, we see that $\lambda = 1$.

Multiplying the series S_4 and S_6 together gives

$$S_4(q)S_6(q) = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \sigma_3(k)\sigma_5(n-k) \right) q^n.$$

Looking at the coefficients of the first expression derived, cancelling a factor of 24 from both sides and taking empty sums as zero, we now see that

$$\sum_{n=1}^{\infty}\left(21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{k=1}^{n-1}\sigma_3(k)\sigma_5(n-k)\right) = \sum_{n=1}^{\infty} 11\sigma_9(n).$$

Hence, for $n \in \mathbb{N}$,

$$21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{k=1}^{n-1}\sigma_3(k)\sigma_5(n-k) = 11\sigma_9(n).$$

This is non-obvious relation which may be a lot harder to prove from first principles or by using other methods. However, for us, it dropped out nicely from our previous work, showing just how interesting and useful modular forms can be.

Chapter 5

Hecke Operators for Γ

In this chapter, we are going to introduce Hecke operators for Γ. Hecke operators are powerful tools that help us to study the spaces of modular forms. In particular, using these operators, we can derive derive some interesting results concerning the spaces $\mathcal{S}_k(\Gamma)$, as well as some classical number theoretic results. The information for this chapter is mainly drawn from two sources, [Apo13, pp. 120–133] and [Sch19, Part III: Hecke Operators].

5.1 Defining Hecke Operators for Γ

We start by defining a few objects which we will use throughout this chapter.

Definition 5.1.1. For $n \in \mathbb{N}$, we define

$$\Pi_n = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2(\mathbb{Z}) \mid a, c \geq 1, ac = n,\ 0 \leq b < c \right\}.$$

Definition 5.1.2 (Slash Operator)**.** Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function, with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) = \{M \in M_n(\mathbb{R}) | \det(M) > 0\}$ and $k \in \mathbb{Z}$. We define the slash operator of weight k acting on f as

$$(f \mid_k \gamma)(z) = (\det \gamma)^{k/2}(cz + d)^{-k} f(\gamma z).$$

Remark 5.1.3.

i) Note for $\gamma \in \Gamma$ and $f \in \mathcal{M}_k(\Gamma)$, we have that

$$f \mid_k \gamma = f,$$

i.e. modular forms are invariant under the slash operator when it is restricted to matrices in Γ.

ii) From the definition of the slash operator, we see that it is linear, i.e.

$$(\lambda f + \mu g) |_k = \lambda f |_k \gamma + \mu g |_k \gamma.$$

iii) The slash operator defines a right action of $GL_n^+(\mathbb{R})$ on the space of meromorphic functions on the upper half plane. The proof is exactly the same as in Proposition 3.1.3.

iv) We have that if f is holomorphic on $\mathbb{H}$, then so is the slash operator, since $cz+d \neq 0$, for $z \in \mathbb{H}$. So, the slash operator also defines a right action of $GL_n^+(\mathbb{R})$ on the space of holomorphic functions on the upper half plane.

Note that for any $n \in \mathbb{N}$, we have a left and right action of Γ on $M(n) = \{A \in M_2(\mathbb{Z}) | \det(A) = n\}$. This follows from the fact that matrices in Γ have determinant one. From this and the above, we can define the so called Hecke operator.

Definition 5.1.4. (Hecke Operator). Let $n \in \mathbb{N}$. We define the Hecke operator of weight n acting on $f \in \mathcal{M}_k(\Gamma)$ as

$$T_n(f)(z) = n^{k/2-1} \sum_{\Gamma\cdot A \,\in\, \Gamma\backslash M(n)} (f |_k A)(z),$$

where here the sum is over a representative of each orbit of the left action of Γ on $M(n)$.[1]

This sum is well defined. Indeed, for each orbit ΓA, if we chose a different representative of ΓA, say $B \in \Gamma A$, then we have that $B = \gamma A$ for some $\gamma \in \Gamma$. But then

$$f |_k B = f |_k \gamma A = f |_k A,$$

since $f \in \mathcal{M}_k(\Gamma)$ and the slash operator defines a right action. Hence, the sum is well-defined.

It would be nice to have a convenient set of representatives to work with. This motivates the following proposition.

Proposition 5.1.5. *We have*

$$M(n) = \bigcup_{\pi \in \Pi_n} \Gamma\pi.$$

Proof. Let $\delta \in \bigcup_{\pi\in\Pi_n} \Gamma\pi$. Then, $\delta = \gamma\pi$, for some $\gamma \in \Gamma$ and $\pi \in \Pi_n$. So,

$$\det(\delta) = \det(\gamma\pi) = \det(\gamma)\det(\pi) = n.$$

Thus, $\delta \in M(n)$.

[1] Zagier in [Zag08] gives a definition which might appeal to a reader more familiar with functions on lattices.

Now, let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n)$. Let us define $g = c/\gcd(a,c)$ and $h = -a/\gcd(a,c)$. Hence, we have that $\gcd(g,h) = 1$. By the Euclidean algorithm, there exists $e, f \in \mathbb{Z}$ such that $eh - fg = 1$. So, by construction, $\delta := \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \Gamma$. Multiplying δ with M, we have

$$\delta M = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae + fc & be + df \\ ag + ch & bg + dh \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ 0 & bg + dh \end{pmatrix} := \begin{pmatrix} i & j \\ 0 & k \end{pmatrix}.$$

Note that $n = \det(M) = det(\delta M) = ik$. We can assume without loss of generality that $i > 0$ and thus $k > 0$, since otherwise we can multiply both sides by $-I$ and absorb the $-I$ on the left hand side into δ, the result of which would still be an element Γ. Finally we have,

$$\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \delta M = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & j \\ 0 & k \end{pmatrix} = \begin{pmatrix} i & kl + j \\ 0 & k \end{pmatrix}.$$

A suitable choice of l, found using division with remainder, gives $0 \leq kl + j < k$. Note that $\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \in \Gamma$. Therefore, as $\begin{pmatrix} i & kl + j \\ 0 & k \end{pmatrix} \in \Pi_n$, we can write $M = \gamma\pi$, for $\gamma \in \Gamma$ and $\pi \in \Pi_n$. □

This shows that every orbit in $\Gamma \backslash M(n)$ has a representative in Π_n. For Π_n to give us a complete set of representatives for this action, we need to check that each distinct element of Π_n, as a representative, gives us a distinct orbit.

Proposition 5.1.6. *For π_1, $\pi_2 \in \Pi_n$, we have*

$$\Gamma\pi_1 = \Gamma\pi_2 \iff \pi_1 = \pi_2.$$

Proof. If $\pi_1 = \pi_2$, then we immediately have that $\Gamma\pi_1 = \Gamma\pi_2$.

Else if $\Gamma\pi_1 = \Gamma\pi_2$, with $\pi_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$, then $g = \gamma h$, for some $\gamma = \begin{pmatrix} g & h \\ i & j \end{pmatrix} \in \Gamma$. Explicitly,

$$\begin{pmatrix} g & h \\ i & j \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} gd & ge + hf \\ id & ie + fj \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

From this we see that $i = 0$, since d $\neq 0$. By considering the determinant of γ, we see that $gj = 1$. This gives that $g = j = \pm 1$. However, we have $c = fj$, with $f, c > 0$. Hence, $g = j = 1$. Thus, we conclude that $a = d$ and $c = f$. Finally, we have $b = hc + e$. But $0 \leq b, e < c$, so $h = 0$ and thus $b = e$. □

In particular, Π_n is a complete set of representatives for the orbit space $\Gamma \backslash M(n)$. Furthermore, this also shows that there are only finitely many orbits in $\Gamma \backslash M(n)$. In conclusion, may write we the weight n Hecke operator acting on $f \in \mathcal{M}_k(\Gamma)$ as

$$T_n(f)(z) = n^{k/2-1} \sum_{\pi \in \Pi_n} (f \mid_k \pi)(z) = n^{k-1} \sum_{1 \leq c \mid n,\, n=ac} \; \sum_{0 \leq b < c} c^{-k} f\left(\frac{az+b}{c}\right).$$

To wrap up this section, we will prove an important lemma that will be useful later.

Lemma 5.1.7. *The set $\{\Gamma\pi \mid \pi \in \Pi_n\}$ is invariant under the right action of Γ.*

Proof. For $\gamma_1 \in \Gamma$ and $\pi_1 \in \Pi_n$, Proposition 5.1.5 tells us that there exist $\gamma_2 \in \Gamma$ and $\pi_2 \in \Pi_n$ such that $\pi_1\gamma_1 = \gamma_2\pi_2$. So we have

$$\Gamma\pi_1\gamma_1 = \Gamma\gamma_2\pi_2 = \Gamma\pi_2.$$

Furthermore, if there is another $\pi_3 \in \Pi_n$ such that $\Gamma\pi_3\gamma_1 = \Gamma\pi_2$, then $\Gamma\pi_1 = \Gamma\pi_3$. But, by Proposition 5.1.6, this holds if and only if $\pi_1 = \pi_3$. This then completes the proof. □

5.2 Properties of Hecke Operators

In this section we will see some interesting properties of Hecke operators. First we will give an important consequence of Lemma 5.1.7.

Theorem 5.2.1. *For $f \in \mathcal{M}_k(\Gamma)$ and $n \in \mathbb{N}$, we have that $T_n(f)$ is weakly modular of weight k.*

Proof. We have $\Gamma \backslash M(n) = \{\Gamma\pi_1, \cdots, \Gamma\pi_n \mid \pi_i \in \Pi_n\}$. So, for $\gamma \in \Gamma$, we have using Lemma 5.1.7 that

$$\begin{aligned} T_n(f) \mid_k \gamma &= n^{k/2-1} \sum_{i=1}^{n} f \mid_k \pi_i \mid_k \gamma = n^{k/2-1} \sum_{i=1}^{n} f \mid_k \pi_i\gamma \\ &= n^{k/2-1} \sum_{i=1}^{n} f \mid_k \tilde{\gamma}_i \pi_{\sigma(i)} = n^{k/2-1} \sum_{j=1}^{n} f \mid_k \pi_j = T_n(f), \end{aligned}$$

for $\tilde{\gamma}_i \in \Gamma$ and $\sigma \in \Sigma_n$ - a permutation of the set $\{1, ..., n\}$. By definition, this shows that $T_n(f)$ is weakly modular of weight k. □

So for $f \in \mathcal{M}_k(\Gamma)$, we have that $T_n(f)$ is holomorphic on $\mathbb{H}$ and weakly modular of weight k. So, $T_n(f)$ is one condition away from being a modular form of weight k. This motivates the next proposition.

Proposition 5.2.2. *Let $f \in \mathcal{M}_k(\Gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n q^n$. For $m \in \mathbb{N}$, we have*

$$T_m(f)(z) = \sum_{n=0}^{\infty} b_n q^n, \text{ with } b_n = \sum_{1 \leq c \mid \gcd(m,n)} c^{k-1} a_{mn/c^2}\ .$$

Proof. As the slash operators act linearly on complex-valued functions, let us first compute the following:

$$\begin{aligned}
m^{k/2-1} \sum_{\gamma \in \Pi_m} q^m \mid_k \gamma &= m^{k/2-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \Pi_m} m^{k/2} c^{-k} e^{2\pi i n((az+b)/c)} \\
&= m^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \Pi_m} c^{-k} e^{2\pi i n b/c} q^{an/c} \\
&= m^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \Pi_m} c^{-k} e^{2\pi i n b/c} q^{mn/c^2} \\
&= m^{k-1} \sum_{1 \le c \mid m} c^{-k} \left(\sum_{0 \le b < c} e^{2\pi i n b/c} \right) q^{mn/c^2}.
\end{aligned}$$

Note that we have

$$\sum_{0 \le b < c} e^{2\pi i n b/c} = \begin{cases} 0 & \text{for } c \nmid n, \\ c & \text{otherwise.} \end{cases}$$

To see this, first suppose that $c \mid n$. Then $n = ck$, for some $k \in \mathbb{N}_0$. In this case,

$$\sum_{0 \le b < c} e^{2\pi i k b} \sum_{0 \le b < c} 1 = c.$$

Next, suppose $c \nmid n$. First note that that $e^{2\pi i n/c}$ is a cth root of unity. Since $c \nmid n$, we have that $e^{2\pi i n/c} \neq 1$. Thus

$$\sum_{0 \le b < c} e^{2\pi i n b/c} = \sum_{0 \le b < c} (e^{2\pi i n/c})^b = 0,$$

since any c th root of unity not equal to one is a solution of the polynomial $1 + x + x^2 + \cdots + x^{c-1}$, using $(x^c - 1)/(x - 1) = 1 + x + x^2 + \cdots + x^{c-1}$.

Hence we conclude from this calculation that

$$m^{k/2-1} \sum_{\gamma \in \Pi_n} q^n \mid_k \gamma = m^{k-1} \sum_{1 \le c \mid \gcd(n,m)} c^{1-k} q^{mn/c^2}.$$

Now we use the linearity of the slash operator to compute,

$$\begin{aligned}
T_m(f)(z) = m^{k/2-1} \sum_{\gamma \in \Pi_m} f \mid_k \gamma = \sum_{n=0}^{\infty} a_n \left(m^{k/2-1} \sum_{\gamma \in \Pi_m} q^n \mid_k \gamma \right) = \\
= \sum_{n=0}^{\infty} \sum_{1 \le c \mid \gcd(m,n)} a_n \left(\frac{m}{c} \right)^{k-1} q^{mn/c^2}.
\end{aligned}$$

Since $c|m$ and $m, c > 0$, we have $m = cd$, for some $d \in \mathbb{N}$. In addition, as $c|n$ with $c \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have $n = ce = me/d$, for some $e \in \mathbb{N}_0$. Hence,

$$\begin{aligned}T_m(f)(z) &= \sum_{n=0}^{\infty} \sum_{1 \leq c | \gcd(m,n)} a_n \left(\frac{m}{c}\right)^{k-1} q^{mn/c^2} = \sum_{e=0}^{\infty} \sum_{1 \leq d | m} d^{k-1}\, a_{me/d}\, q^{de} \\ &= \sum_{n=0}^{\infty} \left(\sum_{1 \leq d | \gcd(m,n)} d^{k-1}\, a_{mn/d^2} \right) q^n,\end{aligned}$$

where we have used the change of variables $n = de$ for the last equality. □

Putting all of our results together we obtain the following:

Corollary 5.2.3. *For $f \in \mathcal{M}_k(\Gamma)$ and $n \in \mathbb{N}$, we have $T_n(f) \in \mathcal{M}_k(\Gamma)$.*

Proof. The proof follows from Remark 5.1.3, Theorem 5.2.1 and Proposition 5.2.2 . □

Remark 5.2.4. Note that for $n \in \mathbb{N}$, we have $T_n : \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma)$. Since the slash operator is linear, we see that T_n is actually a linear map , so $T_n \in \mathrm{End}(\mathcal{M}_k(\Gamma))$.

Now let us see a simple example of computing the action of a Hecke operator explicitly

Example 5.2.5. Let us calculate $T_2(E_4)$. From the definition of Π_2, we see that $a, c \geq 1$ and $ac = 2$. Meaning we only have $(a, c) = (1, 2), (2, 1)$ for possible pairs of a and d. The options for b are determined by the value of c. Hence,

$$\Pi_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

So

$$\begin{aligned}T_2(E_4)(z) =& 2\left(E_4\Big|_4 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + E_4\Big|_4 \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + E_4\Big|_4 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= E_4(z/2)/2 + E_4((z+1)/2)/2 + 8E_4(2z).\end{aligned}$$

By Corollary 5.2.3, we know that $T_2(E_4) \in \mathcal{M}_4(\Gamma) = \mathbb{C}E_4$. Hence, we know that $T_2(E_4) = \lambda E_4$. It will suffice to check the constant coefficient in the q-expansion of $T_2(E_4)$ to determine λ. We know that E_4 has a constant coefficient of one in its q-expansion. From this we deduce that $\lambda = 9$ and so we have $T_2(E_4) = 9E_4$.

Firstly, we note that the above example gives a relation between weight four Eisenstein series composed with fractional linear maps. In addition, we see that E_4 is a eigenvector for T_2. Moreover, since $\mathcal{M}_4(\Gamma)$ is one-dimensional, we conclude that E_4 is actually an eigenvector of T_n, for all $n \in \mathbb{N}$.

Let us now move on to see how Hecke operators act on cusp forms.

Corollary 5.2.6. *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{M}_k(\Gamma)$ *and* $T_m(f)(z) = \sum_{n=0}^{\infty} b_n q^n$. *Then,* $b_0 = a_0\, \sigma_{k-1}(m)$. *In particular,* $a_0 = 0$ *if and only if* $b_0 = 0$.

Proof. From Proposition 5.2.2 we have

$$b_0 = \sum_{1 \le d | \gcd(m,0)} = d^{k-1} a_{m \cdot 0/d^2} = a_0 \sum_{1 \le d | m} d^{k-1} = a_0\, \sigma_{k-1}(m).$$

This gives $b_0 = 0$ if and only if $a_0 = 0$, since $m \in \mathbb{N}$. $\square$

Corollary 5.2.7. *For* $f \in \mathcal{M}_k(\Gamma)$ *and* $n \in \mathbb{N}$, *we have* $f \in \mathcal{S}_k(\Gamma)$ *if and only if* $T_n(f) \in \mathcal{S}_k(\Gamma)$.

Proof. The proof follows immediately from the previous corollary. $\square$

Finally, to conclude this section, we will state some results which tell us how Hecke operators interact with one another.

Theorem 5.2.8. *For* $f \in \mathcal{M}_k(\Gamma)$, *we have the following:*

i) If $\gcd(m, n) = 1$, *we have* $T_n T_m(f) = T_{nm}(f)$.

ii) For $r \in \mathbb{N}$ *and* p *is a prime, then* $T_{p^{r+1}}(f) = T_p T_{p^r}(f) - p^{k-1} T_{p^{r-1}}(f)$.

iii) For all $m, n \in \mathbb{N}$, *we have that* T_m *and* T_n *commute, i.e.* $T_m T_n(f) = T_n T_m(f)$.

Proof. For full details of the proofs of the above claims, see [Apo12, pp. 126–127]. $\square$

5.3 Hecke Eigenforms

In the last chapter, we have seen that E_4 is an eigenfunction of T_n, for all $n \in \mathbb{N}$. Such a function is called a Hecke eigenform.

Definition 5.3.1 (Hecke Eigenform)**.** For $0 \neq f \in \mathcal{M}_k(\Gamma)$, we say that f is a Hecke eigenform if for every $m \in \mathbb{N}$, the function f is an eigenvector of the Hecke operator T_m, i.e. for every $m \in \mathbb{N}$ we have

$$T_m(f) = \lambda_m\, f\ , \quad \text{for some } \lambda_m \in \mathbb{C}.$$

Such an f, with q-expansion $f(z) = \sum_{n \ge 0} a_n q^n$, is said to be a normalised Hecke eigenform if $a_1 = 1$.

The following result is a simple corollary of Proposition 5.2.2.

Corollary 5.3.2. *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{M}_k(\Gamma)$ *and* $T_m(f) = \lambda f$, *for some* $\lambda \in \mathbb{C}$, *then we have* *i)* $a_m = \lambda a_1$.

ii) $a_{mn} = \lambda a_n$ *when* $\gcd(m,n) = 1$.

Proof. For the first claim,

$$\lambda a_1 = b_1 = \sum_{1 \leq d \mid \gcd(m,1)} d^{k-1} a_{m/d^2} = a_m.$$

For the second claim we have that

$$\lambda a_n = b_n = \sum_{1 \leq d \mid \gcd(m,n)} d^{k-1} a_{mn/d^2}.$$

If $\gcd(m,n) = 1$, then the only possible value of d is one. Hence, $\lambda a_n = a_{mn}$. $\square$

Using this result, we can show that the Fourier coefficients of a Hecke eigenform have some very exciting properties.

Proposition 5.3.3. *Let $f \in \mathcal{M}_l(\Gamma)$ be a normalised Hecke eigenform with q-expansion $f(z) = \sum_{n \geq 0} a_n q^n$. Then, the following hold:*

i) For $\gcd(m,n) = 1$, *we have that* $a_{mn} = a_m a_n$.

ii) For a prime p, we have that $a_{p^{r+1}} = a_p a_{p^r} - p^{k-1} a_{p^{r-1}}$.

Proof. Since f is normalised, using Corollary 5.3.2, we have for all $k \in \mathbb{N}$, that $T_k(f) = a_k f$. For $\gcd(m,n) = 1$, using Theorem 5.2.8, we see that

$$a_{mn} f = T_{mn}(f) = T_n T_m(f) = a_m a_n f.$$

Since f is not identically zero, the claim follows (one could also use the second result from Corollary 5.3.3 to prove this). The second claim follows similarly using the second result from Theorem 5.2.8. $\square$

5.3.1 Hecke Eigenform Basis of $\mathcal{S}_k(\Gamma)$

It turns out we can find a basis of $\mathcal{S}_k(\Gamma)$ consisting of Hecke eigenforms. To see this, let us first introduce a hermitian inner product on the spaces of cusp forms.

Definition 5.3.4 (Petersson Inner Product)**.** For $f, g \in \mathcal{S}_k(\Gamma)$ we define the Petersson inner product as

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)} y^k d\mu,$$

where $\Gamma \backslash \mathbb{H}$ refers to the orbit space of the action of Γ on $\mathbb{H}$ and $d\mu = dxdy/y^2$ is the standard hyperbolic measure on $\mathbb{H}$.

The next result will confirm that this is indeed hermitian inner product on $\mathcal{S}_k(\Gamma)$ and will give us the properties we will need from the Petersson inner product to deduce the above claim.

Proposition 5.3.5. *We have*

i) The Petersson Inner Product $\langle \cdot, \cdot \rangle$ defines a hermitian inner product on the space of cusp forms $\mathcal{S}_k(\Gamma)$.

ii) The Hecke operators are self-adjoint with respect to the Petersson inner product, i.e. $\langle T_m(f), g \rangle = \langle f, T_m(g) \rangle$, for every $f, g \in \mathcal{S}_k(\Gamma)$ and $m \in \mathbb{N}$.

Proof. We will not prove this here. See [Gun62, pp. 34–37, 65 and 66] for details. $\square$

Using this result we can now prove the following:

Theorem 5.3.6. *There exists a basis of $S_k(\Gamma)$ consisting of Hecke eigenforms.*

Proof. From Linear Algebra, we know that if an endomorphism of a finite-dimensional inner product space V is self-adjoint, then there exists a basis of V consisting of eigenvectors with real eigenvalues. Hence, in such a case, the endomorphism is diagonalisable. In addition, we have that if $A, B \in End(V)$ are both diagonalisable and A and B commute, i.e. $AB = BA$, then A and B are simultaneously diagonalisable, i.e there exists a basis of V such that both A and B are diagonal with respect to this basis. Using these results, and by appealing to Theorem 5.2.8 and Proposition 5.3.5, completes the proof. $\square$

5.4 An Application of Hecke Operators and Maeda's Conjecture

In this section, we will prove a classical result concerning Ramanujan's τ-function using the theory of Hecke operators. In addition, will also discuss an outstanding conjecture relating to Hecke operators that has drawn a lot of interest over the years.

5.4.1 Ramnujan's $\tau-$Function is Multiplicative

Here, we are going to show a very interesting property of Ramnujan's $\tau-$function, namely that it is a multiplicative.

Example 5.4.1. From Proposition 4.1.17 and Theorem 4.1.18, we see that $\mathcal{M}_{12}(\Gamma) = \mathbb{C}\Delta \oplus \mathbb{C}E_{12}$. By Corollary 5.2.3, we have $T_m(\Delta) \in \mathcal{S}_{12}(\Gamma)$, for every natural number m. So,

$$T_m(\Delta) = \lambda_m \Delta \ , \text{ for } \lambda_m \in \mathbb{C},$$

i.e. Δ is a Hecke eigenform. Note, by definition, that $\tau(1) = 1$. As a result, f is actually a normalised Hecke eigenform. Therefore, by Proposition 5.3.3, for m and n relatively prime, $\tau(mn) = \tau(m)\tau(n)$.

This is a very famous result that was empirically observed in 1916 by Ramanujan, [Ram16]. The result was then proven a year later by Mordell in [Mor17], using a precursor to Hecke operators.

5.4.2 Maeda's Conjecture

Now will will discuss Maeda's conjecture, which is is an outstanding conjecture concerning Hecke operators. First, let us motivate this conjecture with an example.

Example 5.4.2. Let us first calculate the matrix and characteristic polynomial of $T_2 : \mathcal{S}_{36}(\Gamma) \to \mathcal{S}_{36}(\Gamma)$. In this example, we will be using the PARI/GP computer algebra system. The code for the following calculations and results can be found in Appendix D.

From Example 4.1.20, we know that $\dim \mathcal{S}_{36}(\Gamma) = 3$ and that $\mathcal{S}_{36}(\Gamma)$ has a basis $\{\Delta^3, \Delta^2 E_4^3, \Delta E_4^6\}$. Using PARI/GP, we can compute the q-expansions of these basis elements:

$$\begin{aligned}
\Delta^3(z) &= q^3 - 72q^4 + 2484q^5 - 54528q^6 + \cdots, \\
\Delta^2 E_4^3(z) &= q^2 + 672q^3 + 145800q^4 + 9111680q^5 - 233907300q^6 + \cdots, \\
\Delta E_4^6(z) &= q + 1416q^2 + 842652q^3 + 271386688q^4 + 50558976510q^5 \\
&\quad + 5356057835232q^6 + \cdots .
\end{aligned}$$

Given $f(z) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_{36}(\Gamma)$, Corollary 5.2.7 and Proposition 5.2.2 tell us that

$$T_2(f)(z) = \sum_{n=1}^{\infty} b_n q^n, \quad \text{with } b_n = \begin{cases} a_{2n} + 2^{35} a_{n/2} & \text{for } n \text{ even}, \\ a_{2n} & \text{for } n \text{ odd}. \end{cases}$$

Hence, we have that

$$\begin{aligned}
T_2(\Delta^3)(z) &= -72q^2 - 54528q^3 + \cdots, \\
T_2(\Delta^2 E_4^3)(z) &= q + 145800q^2 - 233907300q^3 + \cdots, \\
T_2(\Delta E_4^6)(z) &= 1416q + 34631125056q^2 + 5356057835232q^3 + \cdots .
\end{aligned}$$

We can write each of these in terms of our chosen basis for $\mathcal{S}_{36}(\Gamma)$ quite easily since each element in this basis has a different order a infinity. So, we deduce that

$$\begin{aligned}
T_2(\Delta^3) &= -6144\Delta^3 - 72\Delta^2 E_4^3 + 0\Delta E_4^6, \\
T_2(\Delta^2 E_4^3) &= -331776000\Delta^3 + 144384\Delta^2 E_4^3 + \Delta E_4^6, \\
T_2(\Delta E_4^6) &= -17915904000000\Delta^3 + 34629120000\Delta^2 E_4^3 + 1416\Delta E_4^6.
\end{aligned}$$

Thus, with respect to our basis,

$$T_2 = \begin{pmatrix} -6144 & -331776000 & -17915904000000 \\ -72 & 144384 & 34629120000 \\ 0 & 1 & 1416 \end{pmatrix}.$$

From this, using PARI/GP again, we can calculate the characteristic polynomial of the endomorphism T_2:

$$\begin{aligned} p_2(x) = \det(xI - T_2) = \det &\begin{pmatrix} -6144 - x & -331776000 & -17881274864684 \\ -72 & 144384 - x & 334629120000 \\ 0 & 1 & 1416 - x \end{pmatrix} \\ &= x^3 - 139656x^2 - 59208339456x - 1467625047588864. \end{aligned}$$

Using the PARI/GP function polisirreducible , we see that p_2 is irreducible over $\mathbb{Q}$.

Let L denote the splitting field of p_2 over $\mathbb{Q}$. Firstly, we have $L/\mathbb{Q}$ is normal, since it is the splitting field of a polynomial in $\mathbb{Q}[x]$. We know any field extension of $\mathbb{Q}$ is separable. Hence, we have $L/\mathbb{Q}$ is Galois. Recall that if the discriminant of a cubic is not a square in $\mathbb{Q}$, then the Galois group of the splitting field of the cubic is isomorphic to Σ_3 - symmetric group on three letters. Computing the discriminant using the PARI/GP function poldisc, we see that

$$\Delta_0 = 60603748504919670934480890101760 0.$$

Checking its prime factorisation using the PARI/GP factor function gives

$$606037485049196709344808901017600 = 2^{30} \cdot 3^{10} \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 1259 \cdot 269461929553.$$

Hence, we have that $\Delta_0 \notin \mathbb{Q}^2$. Thus, we conclude that $Gal(L/\mathbb{Q}) \cong \Sigma_3$.

Maeda conjectured that the above properties of the T_2 operator acting on $\mathcal{S}_{36}(\Gamma)$ hold in general:

Conjecture 5.4.3 (Y.Maeda, Conjecture 1.2, [HM97])**.** *The Hecke algebra over $\mathbb{Q}$ of $\mathcal{S}_k(\Gamma)$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to a symmetric group Σ_n (with $n = \dim \mathcal{S}_k(\Gamma)$).*

Or framed more in line with the example above:

Conjecture 5.4.4 (Conjecture 1.1, [GM12])**.** *Let $m > 1$ and let p_m be the characteristic polynomial of the Hecke operator T_m acting on $\mathcal{S}_k(\Gamma)$. Then*

1. *the polynomial p_m is irreducible over $\mathbb{Q}$;*
2. *the Galois group of the splitting field of p_m is the full symmetric group Σ_d, where d is the dimension of $\mathcal{S}_k(\Gamma)$.*

Over the last few years, this conjecture has drawn a lot of attention. Work has been done to computationally verify Maeda's conjecture for larger and larger k and m. A lot of these verifications have been nicely summarised and extended upon in [GM12], where it has been shown for the case when $m = 2$ that the conjecture holds up to $k = 12000$. For the case of $m > 2$, Farmer and James show in [FJ02] that the result holds for $k \leq 2000$ with $m = p < 2000$, where p is a prime.

Chapter 6

Dirichlet Series, L-Functions and Antiderivatives of Modular Forms

In this chapter we will begin by defining Dirichlet series. Then we will associate a Dirichlet series to a given cusp form, its associated L-function. The material presented here on Dirichlet series and L-functions will draw upon [Sch17, Lectures 12–22, pp. 8–9] and [DS05, pp. 200–206]. Using this material, we will then discuss specific antiderivatives of periodic functions, which will play a role in the upcoming chapter on the Eichler-Shimura isomorphism. For this material, we will be following [CS17, pp. 398–406].

6.1 L-Functions Associated to Cusp Forms

In this section, we will introduce Dirichlet series. After introducing these series, we will then define a Dirichlet series associated to a cusp form, its associated L-function . Such functions are of great importance in modern mathematics. For us, they will give an important connection between modular forms and number theory via of Euler products.

Before we begin, we first define a notation we will use throughout this chapter. Given two complex-valued sequences $\{a_n\} \subset \mathbb{C}$ and $\{b_n\} \subset \mathbb{C}$, we say that $a_n = \mathcal{O}(b_n)$ if there exists a constant $C > 0$ and an $N \in \mathbb{N}$, such that for all $n \geq N$,

$$|a_n| \leq C|b_n|.$$

There is also an analogous definition for functions.

6.1.1 Dirichlet Series

We begin by defining the general form of a Dirichlet series.

Definition 6.1.1 (Dirichlet Series)**.** A series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is know as a Dirichlet series, where $\{a_n\}$ is a sequence of complex numbers and $s \in \mathbb{C}$.

The first question we should ask is does this series converge and if so, where?

Proposition 6.1.2. *For a Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$, if $a_n = \mathcal{O}(n^k)$, for some $k \in \mathbb{R}$, then the series converges absolutely in the half-plane $\mathrm{Re}(s) > k+1$.*

Proof. Since $a_n = \mathcal{O}(n^k)$, there exists a constant $C > 0$ and an $N \in \mathbb{N}$, such that for all $n \geq N$, we have that $|a_n| \leq Cn^k$. So,

$$0 \;\leq\; \left|\frac{a_n}{n^s}\right| \;\leq\; C\left|\frac{n^k}{n^s}\right| \;=\; C\left|\frac{1}{n^{s-k}}\right| = \frac{C}{n^{\mathrm{Re}(s)-k}}\ , \text{ for } n \geq N.$$

Hence, by comparison with the series $1/n^{\alpha}$, the Dirichlet series a_n/n^s converges for $\mathrm{Re}(s) - k > 1$, i.e for $\mathrm{Re}(s) > k+1$. $\square$

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ is a Dirichlet series. The above shows that the zeta function converges for $\mathrm{Re}(s) > 1$.

Example 6.1.3. Suppose $a_n = \log(n)$. Then, we know that

$$\lim_{n\to\infty} \frac{\log(n)}{n^{\alpha}} = 0\ , \quad \text{for any } \alpha > 0.$$

This shows that $\log(n) = \mathcal{O}(n^{\epsilon})$, for any $\epsilon > 0$. Hence, the Dirichlet series $\sum_{n=1}^{\infty} \log(n)/n^s$ converges for $\mathrm{Re}(s) > 1$. Note that this series is actually $-\zeta'(s)$, so this makes sense.

6.1.2 L-Functions Associated to Cusp Forms

The notation for the complex sequence $\{a_n\}$ is suggestive. Indeed, given a cusp form $0 \neq f \in \mathcal{S}_k(\Gamma)$ with q-expansion $f(z) = \sum_{n\geq 1} a_n q^n$, we may associate a Dirichlet series to f, know as its L-function, in the following way:

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

To see if and where this function converges, we need to work out how the sequence of Fourier coefficients of f grows.

Lemma 6.1.4. *For $f \in \mathcal{S}_k(\Gamma)$, the function $|f(z)| \cdot \mathrm{Im}(z)^{k/2}$ is bounded on $\mathbb{H}$.*

Proof. Let $z = x+iy$. We claim that $|f(z)| \cdot y^{k/2}$ is invariant under the transformation $z \mapsto \gamma z$, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. To see this, recall that $\mathrm{Im}(gz) = y/|cz+d|^2$. Using this and the fact that f is a modular form of weight k, we have that

$$|f(gz)| \cdot \mathrm{Im}(gz)^{k/2} = |f(z)||cz+d|^k \frac{y^{k/2}}{|cz+d|^k} = |f(z)| \cdot y^{k/2}.$$

So it is sufficient to show that $|f(x+iy)| \cdot y^{k/2}$ is bounded in $\overline{\mathcal{F}}$.

Now, if f has q-expansion $f(z) = \sum_{n\geq 1} a_n q^n$, then,

$$\frac{f(z)}{q} = a_1 + a_2 q + \cdots \to a_1 \text{ as } q \to 0.$$

Therefore, we have for an $\epsilon > 0$, there exists a $\delta > 0$ such that for q satisfying $0 < |q| < \delta$,

$$\left| \frac{f(z)}{q} - c \right| < \epsilon \iff |f(z)| < C \cdot |q|,$$

for some constant $C > 0$. In terms of z, this is equivalent to saying that there exists a $y_0 > 0$, such that for all $y > y_0$, we have that $|f(z)| < C \cdot e^{-2\pi y}$. As a result of this, we conclude that $|f(z)| \cdot y^{k/2}$ is bounded for $y > y_0$, since exponentials beat powers.

For $0 < y \leq y_0$, we have that since $|f(z)| \cdot y^{k/2}$ is a continuous function, it is bounded on the compact set $\overline{\mathcal{F}} \cap \{0 < y \leq y_0\}$. Putting this all together shows that $|f(z)| \cdot y^{k/2}$ is bounded on $\overline{\mathcal{F}}$, which, using the above discussion, gives the claim. $\square$

Proposition 6.1.5. *Let $f \in \mathcal{S}_k(\Gamma)$ with q-expansion $f(z) = \sum_{n=1}^{\infty} a_n q^n$. Then, we have that $a_n = \mathcal{O}(n^{k/2})$.*

Proof. Given $f \in \mathcal{M}_k(\Gamma)$, we have seen that there exists a holomorphic function $\tilde{f} : \mathbb{D} \setminus \{0\} \to \mathbb{C}$, such that $\tilde{f}(e^{2\pi i z}) = f(z)$. We then defined the q-expansion of f as the Laurent series of $\tilde{f}(q)$ about $q = 0$. From complex analysis, we know a formula for calculating the coefficients of a Laurent expansion. So for $0 < r < 1$ and $y = -\log(r)/2\pi$ (so that for $z = x + iy \in \mathbb{H}$, we have $|q| = \left|e^{2\pi i z}\right| = r$), we observe that

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_{|q|=r} \frac{\tilde{f}(q)}{q^{n+1}} dq \right| = \left| \int_{iy}^{1+iy} \frac{f(z)}{e^{2\pi i n z}} dz \right| = \left| \int_0^1 \frac{f(t+iy)}{e^{2\pi i (t+iy) n}} dt \right| \\ &\leq e^{2\pi n y} \int_0^1 |f(t+iy)| dt \leq e^{2\pi n y} \sup_{t \in [0,1]} |f(t+iy)| \leq R \cdot e^{2\pi n y} y^{-k/2}, \end{aligned}$$

where for the last inequality we have used Lemma 6.1.4. The above holds for any $0 < r < 1$ and thus for any $y > 0$. Choosing $y = 1/n$, we get that $|a_n| \leq C \cdot n^{k/2}$, for some some $C > 0$, as required. $\square$

Corollary 6.1.6. *With notation as above, $L(f, s)$ converges absolutely for $\mathrm{Re}(s) > k/2 + 1$.*

Proof. The claim follows immediately from Proposition 6.1.2 and Proposition 6.1.5. $\square$

6.1.3 Euler Products

Using the results in the previous sections, we can actually link the L-function associated to a normalised cuspidal Hecke eigenfom to number theory via Euler products.

Theorem 6.1.7 (Euler Products of Normalised Cuspidal Hecke Eigenforms)**.** *Let $f \in \mathcal{S}_k(\Gamma)$ be a normalised Hecke eigenform with q-expansion $f(z) = \sum_{n\geq 1} a_n q^n$. Then, the L-function associated to f may be written as*

$$L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-2s-1}},$$

for $\mathrm{Re}(s) > k/2 + 1$.

Proof. From Proposition 5.3.3, we know that the Fourier coefficients a_n are multiplicative, which gives that the coefficients of $L(f,s)$ are also multiplicative. Using this and that $L(f,s)$ converges absolutely for $\mathrm{Re}(s) > k/2+1$, we have using [Apo13, Theorem 11.6, p. 230] that

$$L(f,s) = \prod_{p \text{ prime}} (1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots), \tag{6.1}$$

for $\mathrm{Re}(s) > k/2+1$.

Since f is a normalised Hecke eigenform, Proposition 5.3.3 tells us that $a_{p^n} = a_p a_{p^{n-1}} - p^{k-1} a_{p^{n-2}}$, for any prime p. Multiplying both sided of this expression with p^{-ns} and summing over $n \geq 2$ gives

$$\begin{aligned}\sum_{n=2}^{\infty} a_{p^n} p^{-ns} &= p^{-s} a_p \sum_{n=2}^{\infty} a_{p^{n-1}} p^{-(n-1)s} - p^{-2s} p^{k-1} \sum_{n=2}^{\infty} a_{p^{n-2}} p^{-(n-2)s} \\ &= p^{-s} a_p \sum_{n=1}^{\infty} a_{p^n} p^{-ns} - p^{-2s} p^{k-1} \sum_{n=0}^{\infty} a_{p^n} p^{-ns}.\end{aligned}$$

For the last step, we have just changed the index of the sums on the right hand side. Note that all of the series above converge absolutely by comparison with the geometric series, using that $a_n = \mathcal{O}(n^{k/2})$. Now, taking out a common factor by adding and subtracting terms, we see that

$$\left(\sum_{n=0}^{\infty} a_{p^n} p^{-ns}\right)(1 - a_p p^{-s} + p^{k-2s-1}) = 1 + p^{-s} a_p - p^{-s} a_p = 1.$$

Combining this with Equation (6.1) gives the required result. $\square$

We have already seen that Ramnaujan's τ-function is a Hecke eigenform. In particular we have that

$$\begin{aligned}&L(\Delta, s) \\ &= \prod_{p \text{ prime}} \frac{1}{1 - \tau(p) p^{-s} + p^{11-2s}} = \left(\frac{1}{1 + 24p^{-s} + p^{11-2s}}\right)\left(\frac{1}{1 - 252p^{-s} + p^{11-2s}}\right)\cdots.\end{aligned}$$

Now let us see a different example.

Example 6.1.8. It turns out to be rare for the product of two normalised Hecke eigenforms to be again a normalised Hecke eigenform. Such cases are restricted to low dimensional spaces; see [Duk99] for details.

One example of a normalised cuspidal eigenform is ΔE_4, which follows from the fact that $\dim(\mathcal{S}_{16}(\Gamma)) = 1$. Calculating its q-expansion, we see that

$$(\Delta E_4)(z) = \sum_{n=1}^{\infty} b_n q^n,$$

where $b_n = \sum_{k=1}^{n} \tau(k) a_{n-k}$ and $E_4(z) = \sum_{n=0}^{\infty} a_n q^n$. So the q-expansion of ΔE_4 looks like

$$(\Delta E_4)(z) = q + 216q^2 - 3348q^3 + 13888q^4 + \cdots .$$

Hence,

$$L(\Delta E_4, s) = \prod_{p \text{ prime}} \frac{1}{1 - b_p p^{-s} + p^{15-2s}} = \prod_{p \text{ prime}} \frac{1}{1 - \left(\sum_{k=1}^{p} \tau(k) a_{p-k}\right) p^{-s} + p^{15-2s}}$$

$$= \left(\frac{1}{1 - 216p^{-s} + p^{15-2s}}\right)\left(\frac{1}{1 + 3348p^{-s} + p^{15-2s}}\right)\left(\frac{1}{1 - 13888p^{-s} + p^{15-2s}}\right)\cdots .$$

6.1.4 Holomorphic Properties of $L(f, s)$

Let us introduce the Mellin transform, which is similar to the well know Fourier transform. We will use this transformation to prove that $L(f, s)$ extends to an entire function.

Definition 6.1.9 (Mellin Transform)**.** For a function $f : (0, \infty) \to \mathbb{C}$, the Mellin transform of f is given by

$$\mathcal{M}(f, s) = \int_0^{\infty} f(t) t^{s-1} dt,$$

for values of $s \in \mathbb{C}$ where the integral converges.

The following lemma will play a crucial role in proving that $L(f, s)$ extends to an entire function.

Lemma 6.1.10. *Suppose $f : (0, \infty) \to \mathbb{C}$ is continuous and we have that*

i) For every $n \in \mathbb{Z}$, we have $t^n f(t) \to 0$, as $t \to \infty$,

ii) There exists an $m \in \mathbb{R}$ such that $f(t) = \mathcal{O}(t^{-m})$, as $t \to 0$.

Then $\mathcal{M}(f, s)$ converges absolutely to a holomorphic function for $\operatorname{Re}(s) > m$.

Proof. We will not prove this here. One can find an outline of the proof in [Sch17]. $\square$

Using the above, we can show the following:

Theorem 6.1.11. *For $f \in S_k(\Gamma)$, we have that $L(f,s)$ satisfies*

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s),$$

where $\Lambda(f,s) = \mathcal{M}(f(it),s)$. Moreover, $\Lambda(f,s)$ extends to an entire function and satisfies

$$\Lambda(f,k-s) = (-1)^{k/2}\Lambda(f,s).$$

Proof. We apply the Mellin transform to $f(it)$. Note that we have

$$t^m f(it) = \sum_{n=1}^{\infty} \frac{a_n t^m}{e^{2\pi t n}} \to 0 \text{ as } t \to \infty,$$

for any $m \in \mathbb{Z}$. In addition, f satisfies

$$t^k f(it) = i^{-k} f(i/t),$$

which we have seen is bounded as $t \to 0$. Hence, by Lemma 6.1.10, we have $\Lambda(f,s)$ converges to a holomorphic function for $\mathrm{Re}(s) > k$.

Now, we have for $\mathrm{Re}(s) > k/2 + 1$ that

$$\begin{aligned}\Lambda(f,s) &= \int_0^\infty \left(\sum_{n=1}^{\infty} a_n e^{-2\pi t n}\right) t^{s-1} dt = \sum_{n=1}^{\infty} a_n \left(\int_0^\infty t^{s-1} e^{-2\pi t n} dt\right) \\ &= (2\pi)^{-s}\left(\int_0^\infty t^{s-1}e^{-t}\right)\left(\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right) = (2\pi)^{-s}\Gamma(s)L(f,s),\end{aligned}$$

where for the third equality we have used the change of variables $t \mapsto t/(2\pi n)$ and the absolute convergence of $L(f,s)$ justifies the exchange of the sum and integral.

Now computing $\Lambda(f,s)$ explicitly gives

$$\Lambda(f,s) = \int_0^\infty f(it)t^{s-1}dt = \int_0^1 f(it)t^{s-1}dt + \int_1^\infty f(it)t^{s-1}dt.$$

Note, under the change of variables $t \mapsto 1/t$, the first of the two integrals on the right hand side becomes:

$$\begin{aligned}\int_0^1 f(it)\ t^{s-1}dt &= \int_1^\infty f(i/t)\ t^{-s-1}dt = \int_1^\infty f(it)\ i^k\ t^{k-s-1}dt \\ &= \int_1^\infty f(it)\ (-1)^{k/2}\ t^{k-s-1}dt.\end{aligned}$$

So, we have that

$$\Lambda(f,s) = \int_1^\infty f(it)(t^{s-1} + (-1)^{k/2}\ t^{k-s-1})dt.$$

Since we have seen $f(it) = \mathcal{O}(e^{-2\pi t})$, as $t \to \infty$ (cf. the proof of Lemma 6.1.4), we have that the above integral converges for all $s \in \mathbb{C}$ and thus $\Lambda(f, s)$ extends to a entire function.

Finally, we have under the transformation $s \mapsto k - s$ that

$$\Lambda(f, k-s) = \int_1^\infty f(it)(t^{k-s-1} + (-1)^{k/2}\, t^{k-(k-s)-1})dt = (-1)^{k/2}\Lambda(f, s).$$

$\square$

Corollary 6.1.12. *$L(f, s)$ extends to an entire function.*

Proof. Since $\Lambda(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s)$, the function $\Lambda(f, s)$ extends to an entire function and $\Gamma(s)$ has no zeroes in $\mathbb{C}$, it follows that $L(f, s)$ extends to an entire function. $\square$

6.2 Antiderivatives of Modular Forms

In this section we will develop some more theory that will help us tackle the Eichler-Shimura isomorphism.

6.2.1 A Specific Antiderivative

To begin with, we are going to consider antiderivatives of modular forms. Specifically, for $f \in \mathcal{M}_k(\Gamma)$, we will define a function $f^* : \mathbb{H} \to \mathbb{C}$ such that

$$(f^*)^{(k-1)}(z) = \frac{d^{k-1}}{dz^{k-1}} f^*(z) = f(z).$$

So far, we have considered $f \in \mathcal{M}_k(\Gamma)$ with q-expansion/Fourier series given by $f(z) = \sum_{n=0}^\infty a_n q^n$, which comes from the fact that f is a periodic function of period one. In this chapter, we will generalise this slightly by considering holomorphic functions with period $\lambda > 0$. For example, such a function g would have q-expansion/Fourier expansion $g(z) = \sum_{n=0}^\infty a_n q^{n/\lambda}$.

Proposition 6.2.1. *Let $f(z) = \sum_{n=0}^\infty a_n q^{n/\lambda}$ and $g(z) = \sum_{n=0}^\infty b_n q^{n/\lambda}$, with $k \in \mathbb{N}$. Moreover, suppose $a_n = \mathcal{O}(n^\alpha)$ and $b_n = \mathcal{O}(n^\alpha)$, for some $\alpha > 0$. Define the function $f^* : \mathbb{H} \to \mathbb{C}$ by*

$$f^*(z) = a_0 \frac{z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{1-k} \sum_{n=1}^\infty n^{1-k} a_n q^{n/\lambda}.$$

Then, $(f^*)^{(k-1)}(z) = f(z)$. Furthermore, suppose that $f(-1/z) = C(z/i)^k g(z)$, for some $C \in \mathbb{C}^\times$. Then

$$f^*(-1/z) = (-1)^{k-1} C(z/i)^{2-k} g^*(z) + \left(\frac{2\pi i}{\lambda}\right)^{1-k} \sum_{0 \le j \le k-2} \frac{(-2\pi i/\lambda z)^j}{j!} L(f, k-1-j).$$

Proof. Checking the first part of the proposition, we see that

$$
\begin{aligned}
(f^*)^{k-1}(z) = \frac{d^{k-1}}{dz^{k-1}} f^*(z) &= \frac{a_0}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} z^{k-1} + \left(\frac{2\pi i}{\lambda}\right)^{1-k} \frac{d^{k-1}}{dz^{k-1}} \sum_{n=1}^{\infty} n^{1-k} a_n e^{2\pi i n z/\lambda} \\
&= a_0 \frac{(k-1)!}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{1-k} \sum_{n=1}^{\infty} n^{1-k} a_n \left(\frac{2\pi i}{\lambda}\right)^{k-1} n^{k-1} q^{n/\lambda} \\
&= \sum_{n=1}^{\infty} a_n q^{n/\lambda} = f(z).
\end{aligned}
$$

Here we have the right to differentiate the sum term by term as $\sum_{n\geq 1} n^{1-k} a_n e^{2\pi i n/\lambda}$ converges uniformly on $\mathbb{H}$. Indeed, this follows from the bound on the coefficients of f and the Weierstraß M-test.

The rest of the proof is a long exercise in complex analysis. We refer the reader to [CS17, Proposition 11.5.1, pp. 398–400]. □

As a reality check, since later we'll be applying this proposition to $f \in \mathcal{S}_k(\Gamma)$, let us see how Proposition 6.2.1 applies to $f \in \mathcal{M}_k(\Gamma)$.

Example 6.2.2. For a general $f \in \mathcal{M}_k(\Gamma)$, we have

$$
f^*(z) = a_0 \frac{z^{k-1}}{(k-1)!} + (2\pi i)^{1-k} \sum_{n=1}^{\infty} n^{1-k} a_n q^n.
$$

In particular, if $f \in \mathcal{S}_k(\Gamma)$, then f^* has a rather simple form:

$$
f^*(z) = (2\pi i)^{1-k} \sum_{n=1}^{\infty} n^{1-k} a_n q^n.
$$

As specific example, for $\Delta \in \mathcal{S}_{12}(\Gamma)$, we have

$$
\Delta^*(z) = (2\pi i)^{1-k} \sum_{n=1}^{\infty} \tau(n) n^{1-k} q^n.
$$

Note that since τ is multiplicative, the coefficients in the q-expansion of Δ^* are too.

Since $f \in \mathcal{M}_k(\Gamma)$, we have that $f(Rz) = f(-1/z) = z^k f(z) = i^k (z/i)^k f(z)$. So taking $C = i^k$, we have that $f(-1/z) = C(z/i)^k f(z)$. So, the second part of Proposition 6.2.1 gives us that

$$
\begin{aligned}
f^*(-1/z) &= (-1)^{k-1} i^k \left(\frac{z}{i}\right)^{2-k} f^*(z) + (2\pi i)^{k-1} \sum_{0\leq j\leq k-2} \frac{(-2\pi i/z)^j}{j!} L(f, k-1-j) \\
&= z^{2-k} f^*(z) + (2\pi i)^{1-k} \sum_{0\leq j\leq k-2} \frac{(-2\pi i/z)^j}{j!} L(f, k-1-j).
\end{aligned}
$$

We may write the indexed sum on the right hand side of this expression as

$$\sum_{0\leq j\leq k-2} \frac{(-2\pi i/z)^j}{j!} L(f,k-1-j) = \sum_{0\leq j\leq k-2} \frac{(2\pi i)^j}{j!} L(f,k-1-j) \left(\frac{-1}{z}\right)^j.$$

Thinking of this as a polynomial evaluated at $-1/z$, we can make the following definition:

Definition 6.2.3 (Period Polynomial)**.** For $f \in \mathcal{M}_k(\Gamma)$, we define the period polynomial associated to f as

$$P_f(X) = \sum_{0\leq j\leq k-2} \frac{(2\pi i)^j}{j!} L(f,k-1-j) X^j \in \mathbb{C}[X].$$

Remark 6.2.4. Period polynomials are an important topic in their own right. For example, we see that the coefficients of a period polynomial are multiples of integral L-values, which are values that are of interest to number theorists.

So we can relate a modular form of weight k to a polynomial of degree $k-2$ and write

$$f^*(-1/z) = z^{2-k} f^*(z) + (2\pi i)^{1-k} P_f(-1/z).$$

6.2.2 Transformation of f^* Under Γ

As mentioned previously, we are ultimately interested in the case when $f \in \mathcal{S}_k(\Gamma)$. In the previous section, we have seen how $f^*(z)$ transforms under the transformation $z \mapsto -1/z$. So, it is quite natural to wonder how f^* transforms under any $\gamma \in \Gamma$.

Let us assume from now on that $f \in \mathcal{S}_k(\Gamma)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and that $k \geq 12$ is an even integer (so that the elements of $\mathcal{S}_k(\Gamma)$ are not all trivial).

To begin with, let us look in to how f^* transforms under γ when $c = 0$. In this case, we have $ad = 1$. So, we may assume without loss of generality that $a = d = 1$, since $-\gamma$ gives the same action as γ on z. So, we are looking at the transformation $z \mapsto \gamma z = z + b$, for some $b \in \mathbb{Z}$. This is quite a simple transformation, for which we can calculate $f^*(\gamma z)$ explicitly:

$$\begin{aligned} f^*(z+b) &= (2\pi i)^{1-k} \sum_{n=1}^{\infty} n^{1-k} a_n e^{2\pi i n(z+b)} = (2\pi i)^{1-k} \sum_{n=1}^{\infty} n^{1-k} a_n e^{2\pi i n z} e^{2\pi i n b} \\ &= (2\pi i)^{1-k} \sum_{n=1}^{\infty} n^{1-k} a_n e^{2\pi i n z} = f^*(z). \end{aligned}$$

The above calculation covers the case when $c = 0$. Now we assume $c \neq 0$. Moreover, again noting $-\gamma$ gives the same action on z as γ, we may assume $c > 0$. Under this assumption, we make the following definitions in order to simplify upcoming calculations.

Definition 6.2.5. Let $f \in S_k(\Gamma)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c > 0$. Then we define a function $f_\gamma : \mathbb{H} \to \mathbb{C}$ as

$$f_\gamma(z) = f\left(\frac{z-d}{c}\right) = \sum_{n=1}^{\infty} a_n e^{-2\pi i n d/c} q^{n/c},$$

with its associated L-function

$$L(f_\gamma, s) = \sum_{n=1}^{\infty} e^{-2\pi i n d/c} \frac{a_n}{n^s}.$$

Example 6.2.6. If $\gamma = R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $f_R(z) = f(z)$ and $L(f_\gamma, s) = L(f, s)$, since $d = 0$ and $c = 1$.

Now using that f is is a cusp form, we can find a useful relation between f_γ and $f_{\gamma^{-1}}$.

Lemma 6.2.7. *If $f \in \mathcal{S}_k(\Gamma)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c > 0$. Then $f_\gamma(-1/z) = z^k f_{\gamma^{-1}}(z)$.*

Proof. Here, will use slash operators and the fact that they define a right action on holomorphic functions on $\mathbb{H}$. Note that

$$f_\gamma(z) = f\left(\frac{z-d}{c}\right) = c^{k/2} f\Big|_k \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix},$$

$$f_{\gamma^{-1}}(z) = f\left(\frac{z+a}{c}\right) = c^{k/2} f\Big|_k \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}.$$

In particular, we may write

$$f_\gamma(-1/z) = z^k f \mid_k \begin{pmatrix} 1 & -d \\ -0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have the following decomposition of γ:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, since $f \in \mathcal{S}_k(\Gamma)$, it is weakly modular of weight k, i.e. $f|_k\gamma = f$, for any $\gamma \in \Gamma$. Using the above, we have that

$$f_\gamma(-1/z) = z^k f\Big|_k \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = z^k f\Big|_k \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} = z^k f_{\gamma^{-1}}(z).$$

□

Remark 6.2.8. Note that $L(f_\gamma, s)$, as defined above, is absolutely convergent for $\mathrm{Re}(s) > k/2 + 1$. This follows from Proposition 6.1.2, using that $a_n e^{-2\pi i n d/c} = \mathcal{O}(n^{k/2})$. Moreover, one can show that $L(f_\gamma, s)$ is an entire function, in particular, the series converges for all $s \in \mathbb{C}$. The proof follows a similar path to the proof of Theorem 6.1.11 except one makes use of Lemma 6.2.7 rather that the property $f(-1/z) = z^k f(z)$, for $f \in \mathcal{S}_k(\Gamma)$.

Using this result, we can give an explicit representation of how f^* transforms under any $\gamma \in \Gamma$.

Proposition 6.2.9. *Let $f \in \mathcal{S}_k(\Gamma)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c > 0$. Then we have*

$$f^*(z) = (cz+d)^{k-2} f^*\left(\frac{az+b}{cz+d}\right) + (2\pi i)^{1-k} \sum_{0 \le j \le k-2} \frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j)(cz+d)^j.$$

Proof. We assume k is even, otherwise $\mathcal{S}_k(\Gamma)$ would be trivial. From Lemma 6.2.7, we know that $f_\gamma(-1/z) = z^k f_{\gamma^{-1}}(z)$. So, applying the second part of Proposition 6.2.1 with $C = i^k$ and $\lambda = c$ gives

$$f_\gamma^*(-1/z) = z^{2-k} f_{\gamma^{-1}}^*(z) + \left(\frac{2\pi i}{c}\right)^{1-k} \sum_{0 \le j \le k-2} \frac{(-2\pi i/cz)^j}{j!} L(f_\gamma, k-1-j).$$

Now, using a change of variables $z \mapsto -1/z$ in the above yields

$$\begin{aligned} f_\gamma^*(z) &= (-1/z)^{2-l} f_{\gamma^{-1}}^*(-1/z) + \left(\frac{2\pi i}{c}\right)^{1-k} \sum_{0 \le j \le k-2} \frac{(-2\pi i/c(-1/z))^j}{j!} L(f_\gamma, k-1-j) \\ &= z^{k-2} f_{\gamma^{-1}}^*(-1/z) + \left(\frac{2\pi i}{c}\right)^{1-k} \sum_{0 \le j \le k-2} \frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j) z^j. \end{aligned}$$

By definition, we have we have $f_{\gamma^{-1}}(z) = f\left(\frac{z+a}{c}\right)$, which gives $f^*(\frac{z+a}{c}) = f_{\gamma^{-1}}^*(z)$. So

$$f_{\gamma^{-1}}^*(-1/z) = f^*\left(\frac{-1/z+a}{c}\right) = f^*\left(\frac{az-1}{cz}\right) = f^*\left(\frac{a\frac{z-d}{c}+b}{c\frac{z-d}{c}+d}\right).$$

Also, we have that $f^*(\frac{z-d}{c}) = f_\gamma^*(z)$. Substituting this into the above gives:

$$f^*\left(\frac{z-d}{c}\right) = z^{k-2} f^*\left(\frac{a\frac{z-d}{c}+b}{c\frac{z-d}{c}+d}\right) + \left(\frac{2\pi i}{c}\right)^{1-k} \sum_{0 \le j \le k-2} \frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j) z^j.$$

Now consider a change of variables $z \mapsto cz + d$. This gives

$$\begin{aligned} f^*(z) = (cz+d)^{k-2} & f^*\left(\frac{az+b}{cz+d}\right) \\ &+ \left(\frac{2\pi i}{c}\right)^{1-k} \sum_{0 \le j \le k-2} \frac{(2\pi i/c)^j}{j!} (z+d/c)^j L(f_\gamma, k-1-j)(cz+d)^j. \end{aligned}$$

$\square$

Chapter 7

Eichler-Shimura Isomorphism

In the final chapter of this project, we will discuss the Eichler-Shimura isomorphism. This isomorphism relates spaces of cusp forms to certain cohomology groups of Γ. The content of this chapter will mainly follow [CS17, pp. 406–414], which in our opinion, gives a more gentle introduction into this theory.

7.1 Eichler Cohomology

Cohomology has its origins in topology. Intuitively, the dth cohomology group measures the number of d–dimensional 'holes' in a topological space. For examples the torus, T, has two one–dimensional holes (think of two different circles on the torus that you cannot shrink to a point) and one two–dimensional hole (the hole at the centre bounded by the surface). The cohomology groups of the torus with coefficients in $\mathbb{Z}$ are

$$H^1(T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \text{ and } H^2(T;\mathbb{Z}) \cong \mathbb{Z}.$$

Here you can see that the rank of the abelian group is the same as the number of 'holes', which lines up with the intuitive picture. A good book for further information on this subject is [Hat00].

It turns out that cohomology can also be useful for studying other objects. For example, one can use cohomology to study groups, but what does it mean for a group to have 'holes'? Regardless of the answer, cohomology is a powerful tool that is used a lot in modern mathematics to examine such objects and was even used in the proof of Fermat's last theorem. Cohomology comes with a lot of machinery that one can exploit, so it is useful to try and link objects to cohomology for this reason.

7.1.1 The Γ-Module $V_m(F)$

Let F be a field extension of $\mathbb{Q}$, usually we will be working with $F = \mathbb{R}$ or $\mathbb{C}$. For an even integer $m \geq 4$, we define the F-vector space $V_m(F)$ to be the set of polynomials in $F[z]$ of degree at most $m-2$.

Recall that we have previously defined the slash operator, which gave us a right action of $GL_n^+(\mathbb{R})$ on the space of holomorphic functions on the upper half plane. In our new setting, we see that the weight $2-m$ slash operator, $|_{2-m}$, which we now denote $|$, defines a right action of Γ on $V_m(F)$:

$$(P \mid \gamma)(z) = (cz+d)^{m-2} P\left(\frac{az+b}{cz+d}\right), \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } P \in V_m(F).$$

This is well-defined since $\mathbb{Z} \subset \mathbb{Q} \subseteq F$. Furthermore, for $P(z) = a_0 + a_1 z + \cdots + a_{m-2}\, z^{m-2}$:

$$\begin{aligned}(P \mid \gamma)(z) &= (cz+d)^{m-2}\Big(a_0 + a_1\left(\frac{az+b}{cz+d}\right) + \cdots + a_{m-2}\left(\frac{az+b}{cz+d}\right)^{m-2}\Big) \\ &= a_0(cz+d)^{m-2} + a_1(az+b)(cz+d)^{m-3} + \cdots + a_{m-2}(az+b)^{m-2}.\end{aligned}$$

Hence, $P \mid \gamma \in V_m(F)$ since we see that it is of degree at most $m-2$. Furthermore, we have already seen for $\gamma, \delta \in \Gamma$, that $P \mid \gamma \mid \delta = P \mid \gamma\delta$. So, the slash operator does indeed define a right action of Γ on $V_m(F)$.

Next, for a group G, we will introduce so called G-modules.

Definition 7.1.1 (G-Module)**.** A right G-module is an abelian group M with a right G-action on M, such that the action is linear in the following sense:

$$(m+n) \cdot g = m \cdot g + n \cdot g\ , \quad \text{for all } m, n \in M \text{ and } g \in G.$$

We have shown above that the slash operator defines a right action on $V_m(F)$. Furthermore, we have seen in our discussion on Hecke operators that the slash operator acts linearly. Hence, we have that $V_m(F)$ is a Γ-module.

7.1.2 Cohomology Groups of Γ

Now we can begin by defining the cohomology groups we will be working with. First we define define a cochain complex. Fix an even integer $m \geq 4$ and a field F, with $char(F) = 0$. Then we define

$$C^n(\Gamma, V_m(F)) = \{f : \Gamma^n \to V_m(F)\}.$$

Here, $f : \Gamma^n \to V_m(F)$ refers to any function between these two sets. Note for $n = 0$, we have that $\Gamma^0 = \{e\}$. So, $C^0(\Gamma, V_m(F)) = \{f : \Gamma^0 \to V_m(f),\ e \mapsto P \in V_m(F)\}$. Some interesting functions to keep in mind are maps of the form

$$f : \Gamma \to V_m(F), \ \ \gamma \mapsto P|\gamma - P, \text{ for some } P \in V_m(F).$$

The set $C^n(\Gamma, V_m(F))$ forms an abelian group under addition. Moreover, since $V_m(F)$ is a F-vector space, $C^n(\Gamma, V_m(F))$ is also an F-vector space. In particular, $C^0(\Gamma, V_m(F)) \cong V_m(f)$. We call $C^n(\Gamma, V_m(F))$ the nth cochain space (or group).

Using this, we can define a cochain complex $(C^*(\Gamma, V_m(F)), \delta^*)$ by

$$0 \longrightarrow C^0(\Gamma, V_m(F)) \xrightarrow{\delta^0} C^1(\Gamma, V_m(F)) \xrightarrow{\delta^1} C^2(\Gamma, V_m(F)) \xrightarrow{\delta^2} \cdots,$$

where, for $n \in \mathbb{N}$, we have that $\delta^n : C^n(\Gamma, V_m(F)) \to C^{n+1}(\Gamma, V_m(F))$ is an F-linear map such that $\delta^n \circ \delta^{n-1} = 0$. In this project, we will be focusing the maps δ^0 and δ^1. We define these as

$$\delta^0(e \mapsto P)(\gamma) = P|\gamma - P \text{ and } \delta^1(\gamma \mapsto f(\gamma))(\gamma_1, \gamma_2) = f(\gamma_1)|\gamma_2 - f(\gamma_1\gamma_2) + f(\gamma_2).$$

Since $V_m(F)$ is a Γ-module, we see that both of these maps are F-linear. Furthermore, we see that $\delta^1 \circ \delta^0 = 0$. Indeed,

$$\begin{aligned}(\delta^1 \circ \delta^0)(e \to P)(\gamma_1, \gamma_2) &= \delta^1(\gamma \mapsto (P|\gamma - P))(\gamma_1, \gamma_2) \\ &= (P|\gamma_1 - P)|\gamma_2 - (P|(\gamma_1\gamma_2) - P) + (P|\gamma_2 - P) \\ &= P|(\gamma_1\gamma_2) - P|\gamma_2 - P|(\gamma_1\gamma_2) + P + P|\gamma_2 - P = 0.\end{aligned}$$

In particular, this shows that $\mathrm{im}(\delta^0) \leq \ker(\delta^1)$, i.e. $\mathrm{im}(\delta^0)$ is a subspace of $\ker(\delta^1)$.

Let $Z^1(\Gamma, V_m(F)) = \ker(\delta^1)$ and $B^1(\Gamma, V_m(F)) = \mathrm{im}(\delta^0)$. By the above, we are justified in defining the first cohomology group of Γ as

$$H^1(\Gamma, V_m(F)) := Z^1(\Gamma, V_m(F))/B^1(\Gamma, V_m(F)).$$

As sets, we have

$$\begin{aligned}Z^1(\Gamma, V_m(F)) &= \{f : \Gamma \to V_m(K) \mid f(\gamma_1\gamma_2) = f(\gamma_1)|\gamma_2 + f(\gamma_2), \text{ for all } \gamma_1, \gamma_2 \in \Gamma\}, \\ B^1(\Gamma, V_m(F)) &= \{f : \Gamma \to V_m(F) \mid f(\gamma) = P|\gamma - P, \text{ for } P \in V_m(K)\}.\end{aligned}$$

An element of $Z^1(\Gamma, V_m(F))$ is known as a cocycle. Elements of $B^1(\Gamma, V_m(F))$ are called coboundaries.

Remark 7.1.2. We can similarly define the zero cohomology group as

$$H^0(\Gamma, V_m(K)) = \ker(d^0)/\mathrm{im}(d^{-1}) \cong \{P \in V_m(K) \mid P|\gamma = P, \text{ for all } \gamma \in \Gamma\} = V_m(F)^\Gamma,$$

which is the set of fixed points of $V_m(F)$ under the right action of the slash operator.

7.1.3 Eichler Cohomology

In this section, we will define a subspace of $H^1(\Gamma, V_m(F))$, namely the first Eichler cohomology group, denoted $\overline{H}^1(\Gamma, V_m(F))$.

Definition 7.1.3 (Cuspidal Cocycle)**.** Let $f : \Gamma \to V_m(F)$ be a cocycle, i.e. $f \in Z^1(\Gamma, V_m(F))$. Then we say that f is a cuspidal cocycle if $\deg(f(T)) < k - 2$. The set of cuspidal cocycles is denoted by $\overline{Z}^1(\Gamma, V_m(F))$.

Having defined this new object, let us now see a family of examples.

Example 7.1.4. We have already see that the functions

$$f : \Gamma \to V_m(F), \ \ \gamma \mapsto P|\gamma - P, \text{ for some } P \in V_m(F)$$

are cocycles (since they lie in the image of δ^0, which is contained in the kernel of δ^1). Evaluating $f(\gamma)$ at $\gamma = T$ gives

$$f(T) = \Big(a_{m-2}\,(z+1)^{m-2} + \cdots + a_0\Big) - \Big(a_{m-2}\,z^{m-2} + \cdots + a_0\Big).$$

From this we can see that the leading term will be killed off, giving that $\deg(f(T)) < k-2$. Hence, $f \in \overline{Z}^1(\Gamma, V_m(\Gamma))$. In particular, this shows that $B^1(\Gamma, V_m(F)) \subseteq \overline{Z}^1(\Gamma, V_m(\Gamma))$.

Remark 7.1.5. For the case when $F = \mathbb{R}$, the only other cuspidal cocycles are ones generated by cusp forms, see Lemma 7.2.1. This follows by considering the dimension of $\overline{Z}^1(\Gamma, V_m(\mathbb{R}))$, see Lemma 7.2.3.

There are a few things to note. Firstly, $\overline{Z}^1(\Gamma, V_m(F))$ forms subspace of $Z^1(\Gamma, V_m(F)$. Secondly, the above shows that $B^1(\Gamma, V_m(F))$ is a subspace of $\overline{Z}^1(\Gamma, V_m(F))$. Therefore, we can again consider the quotient group and define the first Eichler cohomology groups.

Definition 7.1.6 (Eichler Cohomology)**.** For a given even integer $m \geq 4$, we define the first Eichler cohomology group of Γ as

$$\overline{H}^1(\Gamma, V_m(F)) = \overline{Z}^1(\Gamma, V_m(F))/B^1(\Gamma, V_m(F)).$$

7.2 Eichler-Shimura Isomorphism

In this section we will give an isomorphism between $\overline{H}^1(\Gamma, V_k(\mathbb{R}))$ and $\mathcal{S}_k(\Gamma)$ called the Eichler-Shimura isomorphism. Let us again assume that $k \geq 12$ is an even integer. First let us state a couple of results which we will need.

Lemma 7.2.1. *Suppose $f \in \mathcal{S}_k(\Gamma)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have the following:*

i) There exists a $P_\gamma \in V_k(\mathbb{C})$ with

$$f^*|\gamma - f^* = P_\gamma\ , \ \textit{i.e.}\ \ f^*|\gamma - f^* \in V_k(\mathbb{C}).$$

ii) The functions

$$\begin{aligned} \phi_{\mathbb{C}}(f)(\gamma) &= f^*|\gamma - f^* = P_\gamma \in C^1(\Gamma, V_k(\mathbb{C})), \\ \phi_{\mathbb{R}}(f)(\gamma) &= \mathrm{Re}(f^*|\gamma - f^*) = \mathrm{Re}(P_\gamma) \in C^1(\Gamma, V_k(\mathbb{R})), \end{aligned}$$

are cuspidal cocycles. Furthermore, their cohomology class in their respective cohomology groups is independent of the choice of $(k-1)$st antiderivative of f.

Remark 7.2.2. As we have already mentioned, since k is even, slashing f^* with $-\gamma$ will give the same result as slashing f^* with γ (since the slash operators is of weight $2-k$, which is even, and $-\gamma$ gives the same action on z as γ). Hence, we have that

$$\phi_{\mathbb{C}}(f)(-\gamma) = \phi_{\mathbb{C}}(f)(\gamma) \ \text{ and } \ \phi_{\mathbb{R}}(f)(-\gamma) = \phi_{\mathbb{R}}(f)(\gamma).$$

Proof. We may assume without loss of generality that $c \geq 0$. If $c > 0$, then the first claim follows from Proposition 6.2.9. If $c = 0$, then we have seen that $f^*|\gamma - f^* = 0$, so here take $P_\gamma = 0$ (after showing that $\phi_{\mathbb{C}}(f)$ and $\phi_{\mathbb{R}}(f)$ are cocycles, this will show that they are cuspidal cocycles). This proves the first claim.

Note, $\phi_{\mathbb{C}}(f)$ is a cocycle:

$$\begin{aligned}\delta^1(\phi_{\mathbb{C}}(f))(\gamma_1,\gamma_2) &= (f^*|\gamma_1 - f^*)|\gamma_2 - (f^*|\gamma_1\gamma_2 - f^*) + f^*|\gamma_2 - f^* \\ &= f^*|\gamma_1\gamma_2 - f^*|\gamma_2 - f^*|\gamma_1\gamma_2 + f^* + f^*|\gamma_2 - f^* = 0.\end{aligned}$$

Now, since an indefinite integral is unique up to a constant on a connected domain, we have that if f' is another $(k-1)$st antiderivative of f, then $f' = f^* + Q$, for some $Q \in V_k(\mathbb{C})$. So

$$[f'|\gamma - f'] = [(f^*+Q)|\gamma - (f^*+Q)] = [(f^*|\gamma - f^*) + (Q|\gamma - Q)] = [f^*|\gamma - f^*],$$

since $Q|\gamma - Q = \delta^0(e \mapsto Q) \in B^1(\Gamma, V_m(\mathbb{C}))$. This shows that the cohomology class of $\phi_{\mathbb{C}}(f)$ is independent of the choice of a $(k-1)$st antiderivative.

For $\gamma \in \Gamma = SL_2(\mathbb{Z})$, the matrix γ has real entries. In particular,

$$\mathrm{Re}(P|\gamma) = \mathrm{Re}(P)|\gamma,$$

with $\gamma \in \Gamma$ and $P \in V_k(\mathbb{C})$. Using this we have that

$$\begin{aligned}\delta^1(\phi_{\mathbb{R}}(f))(\gamma_1,\gamma_2) &= \mathrm{Re}((f^*|\gamma_1 - f^*))|\gamma_2 - \mathrm{Re}(f^*|\gamma_1\gamma_2 - f^*) + \mathrm{Re}(f^*|\gamma_2) - f^*) \\ &= \mathrm{Re}((f^*|\gamma_1 - f^*)|\gamma_2 - f^*|\gamma_1\gamma_2 + f^* + f^*|\gamma_2 - f^*) = 0\ .\end{aligned}$$

From this we conclude $\phi_{\mathbb{R}}(f)$ is a cocycle. The same reasoning as for $\phi_{\mathbb{C}}(f)$ shows that the cohomology class of $\phi_{\mathbb{R}}(f)$ is independent of the choice of a $(k-1)$st antiderivative of f. □

Lemma 7.2.3. *We have that* $\dim_{\mathbb{R}}(\mathcal{S}_k(\Gamma)) = \dim_{\mathbb{R}}(\overline{H}^1(\Gamma, V_k(\mathbb{R}))$.

Proof. This takes quite a lot of work to prove, mostly consisting of multiple exercises in linear algebra. For details, see [CS17, Section 11.8, pp. 410–414]. □

With all of this, we can finally state and prove the result that goes by the name of the Eichler-Shimura isomorphism:

Theorem 7.2.4 (Eichler-Shimura Isomorphism). *Let $k \in \mathbb{N}$. Then we have*

$$S_k(\Gamma) \cong \overline{H}^1(\Gamma, V_k(\mathbb{R})),$$

as $\mathbb{R}$ vector spaces via the isomorphism

$$\overline{\phi}_{\mathbb{R}} : \mathcal{S}_k(\Gamma) \to \overline{H}^1(\Gamma, V_k(\mathbb{R})), \quad f \mapsto [\phi_{\mathbb{R}}(f)].$$

Here, $[\cdot]$ refers to taking the cohomology class in $\overline{H}^1(\Gamma, V_k(\mathbb{R}))$.

Proof. By Proposition 7.2.1, the map $\overline{\phi}_{\mathbb{R}}$ is well-defined. Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in S_k(\Gamma)$. Then,

$$\begin{aligned}\overline{\phi_{\mathbb{R}}}(\lambda f + \mu g)(\gamma) &= [\mathrm{Re}((\lambda f + \mu g)^*|\gamma - (\lambda f + \mu g)^*)] = [\mathrm{Re}(\lambda(f^*|\gamma - f^*) + \mu(g^*|\gamma - g^*))] \\ &= \lambda[\mathrm{Re}(f^*|\gamma - f^*)] + \mu[\mathrm{Re}(g^*|\gamma - g^*)] = \lambda\overline{\phi}_{\mathbb{R}}(f) + \mu\overline{\phi}_{\mathbb{R}}(g).\end{aligned}$$

So $\overline{\phi}_{\mathbb{R}}$ is an $\mathbb{R}$-linear map.

Let $0 \neq f \in \mathcal{S}_k(\Gamma)$ such that $\overline{\phi}_{\mathbb{R}}(f)(\gamma) = [\mathrm{Re}(f^*|\gamma - f^*)] = [0]$. So $\mathrm{Re}(f^*|\gamma - f^*) \in \mathrm{im}(\delta^0)$, i.e. $\mathrm{Re}(f^*|\gamma - f^*) = P|\gamma - P$, for some $P \in V_k(\mathbb{R})$. We have already seen that $f^*|T = f^*$. So taking $\gamma = T$, we see that

$$0 = \mathrm{Re}(f^*|T - f^*) = P|T - P = P(z+1) - P(z).$$

We can write $P(z) = \sum_{n=0}^{m} a_n z^n$, for some $1 \leq m \leq k-2$ with $a_m \neq 0$. Looking at the coefficient of z^{m-1} of $P(z+1) - P(z)$, we see that

$$\binom{m}{m-1} a_m + a_{m-1} - a_{m-1} = m \cdot a_m = 0.$$

Since $a_m \neq 0$, we conclude $m = 0$. Hence, P is a constant polynomial, i.e. $P = C \in \mathbb{R}$. Therefore, we may split $f^*|\gamma - f^*$ into its real and imaginary parts as follows:

$$(f^*|\gamma - f^*)(z) = (C|\gamma - C) + iQ_\gamma(z) = C\Big((cz+d)^{k-2} - 1\Big) + iQ_\gamma(z),$$

for some $Q_\gamma \in V_k(\mathbb{R})$. For $c > 0$, we know from Proposition 6.2.9 that

$$(f^*|\gamma - f^*)(z) = -(2\pi i)^{1-k} \sum_{0 \leq j \leq k-2} \frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j)(cz+d)^j.$$

Thus, we have

$$C((cz+d)^{k-2} - 1) + iQ_\gamma(z) = -(2\pi i)^{1-k} \sum_{0 \leq j \leq k-2} \frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j)(cz+d)^j.$$

Note that we can write any polynomial in $V_k(\mathbb{R})$ in terms of $\{(cz+d)^j | j = 0, 1, ..., k-2\}$. One can see this using Taylor series. Hence, $\{(cz+d)^j | j = 0, 1, ..., k-2\}$ forms a basis of

$V_k(\mathbb{R})$. Thus, the set $\{(cz+d)^j|j=0,1,...,k-2\} \cup \{i(cz+d)^j|j=0,1,...,k-2\}$ forms a basis of $V_k(\mathbb{C})$. Now, comparing coefficients in the above, keeping in mind these bases, we see, ignoring the $j=0$ and $j=k-2$ terms on the right hand side of the above, that the other terms in the right hand side must form part of the expression of $iQ_\gamma(z)$. Hence,

$$-(2\pi i)^{1-k} \sum_{1\leq j\leq k-3} \frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j)(cz+d)^j \in (\mathbb{R}i)[z]. \tag{7.1}$$

Note that k is even, otherwise $f=0$. Hence, $(2\pi i)^{1-k} \in \mathbb{R}i$, since $1-k$ is odd. So, $-(2\pi i)^{1-k} = \alpha i$, for some $\alpha \in \mathbb{R}$. So for (7.1) to hold, we must have that

$$\frac{(2\pi i/c)^j}{j!} L(f_\gamma, k-1-j) \in \mathbb{R},$$

for $j=1,2,\cdots,k-3$. Looking at the $j=1$ term, we see that

$$\frac{2\pi i}{c} L(f_\gamma, k-2) \in \mathbb{R}.$$

This can only happen if $L(f_\gamma, k-2) \in \mathbb{R}i$. By assumption, $0 \neq f \in \mathcal{S}_k(\Gamma)$. By the Proposition 4.1.17 and Theorem 4.1.18, we must have $k \geq 12$. So,

$$k \geq 12 \implies k > 6 \implies k/2 > 3 \implies k-2 > k/2+1.$$

As result, we have that $L(f_\gamma, k-2)$ converges absolutely (cf. Remark 6.2.8). Now, using the definition of $L(f_\gamma, s)$ and that $L(f_\gamma, k-2) \in \mathbb{R}i$, we have

$$L(f_\gamma, k-2) + \overline{L(f_\gamma, k-2)} = \sum_{n=1}^{\infty} e^{-2\pi i n d/c} \frac{a_n}{n^{k-2}} + \sum_{n=1}^{\infty} e^{2\pi i n d/c} \frac{\overline{a_n}}{n^{k-2}} = 0. \tag{7.2}$$

Now rephrasing this, let us define the following function:

$$g : \mathbb{R} \to \mathbb{C}, \ \ x \mapsto g(x) = \sum_{n=1}^{\infty} e^{-2\pi i n x} \frac{a_n}{n^{k-2}} + \sum_{n=1}^{\infty} e^{2\pi i n x} \frac{\overline{a_n}}{n^{k-2}}.$$

We have that g is a continuous function on $\mathbb{R}$. To see this, note from Corollary 6.1.6 that we have for $L(f, k-2) = \sum_{n\geq 1} a_n/n^{k-2}$ converges absolutely for $k \geq 12$. Hence, by the Weierstraß M-test, we have that the series defining g converges uniformally on $\mathbb{R}$. Moreover, since the summands are continuous, we conclude that g is continuous on $\mathbb{R}$.

Note that given $c/d \in \mathbb{Q}$, assumed to be in its simplest form, i.e. $\gcd(c,d)=1$, the Euclidean algorithm gives $a, b \in \mathbb{Z}$, such that $ad-bc=1$. Hence, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Therefore, using Equation (7.2), we have that $g(x)=0$, for every $x \in \mathbb{Q}$. From this, we must have $g(x)=0$, for all $x \in \mathbb{R}$. To see this suppose, suppose there exists a $y \in \mathbb{R}$ such that $g(y) \neq 0$. Since g is continuous, there exists an $\epsilon > 0$ such that for all $x \in B_\epsilon(y)$, we have $g(x) \neq 0$. However, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational $x \in B_\epsilon(y)$, which gives a contradiction since $g(x)$ vanishes for all $x \in \mathbb{Q}$.

The function g represents an absolutely convergent Fourier series which vanishes on $\mathbb{R}$. From this, we must have that the Fourier coefficients are zero, so $a_n = 0$, for all $n \geq 1$. This means that $f = 0$, which gives us a contradiction. Hence, $\ker(\overline{\phi}_{\mathbb{R}}) = 0$, showing $\phi_{\mathbb{R}}$ is injective.

Since $\overline{\phi}_{\mathbb{R}}$ is an injective $\mathbb{R}$-linear map, $\mathrm{im}(\overline{\phi}_{\mathbb{R}})$ embeds as a subspace of $\overline{H}^1(\Gamma, V_k(\mathbb{R})$. Thus to prove the claim, it is sufficient to show that $\dim_{\mathbb{R}}(\overline{H}^1(\Gamma, V_k(\mathbb{R})) = \dim_{\mathbb{R}}(S_k(\Gamma))$. Therefore, Lemma 7.2.3. gives the required surjectivity and thus we conclude that $\overline{\phi}_{\mathbb{R}}$ is an isomorphism of $\mathbb{R}$-vector spaces. □

This result is very interesting. It gives us a bijection between the space of cusp forms of a particular weight and an Eichler cohomology group. As mentioned at the beginning of the chapter, group cohomology has some very powerful machinery associated to it, so it is compelling to associate the spaces of cusp forms with such structures.

Looking at the actual isomorphism, we see that the Eichler-Shimura map contains information about the values of the L-function $L(f_\gamma, s)$, which is a transformed/twisted version of $L(f, s)$. As we have mentioned previously, L-values are of great interest to mathematicians. L-functions are analytic objects that often contain exciting arithmetic information at special 'critical' points, for example at integer values. A classic example of this would be that because the Riemann zeta function, $\zeta(s)$, has a a pole at $s = 1$, gives that there are infinitely many primes. Hence, it is worthwhile investigating such critical points and the Eichler Shimura isomorphism can help with this. Indeed it is a major tool used in the proof of a theorem by Manin and Shimura concerning properties of L-values; see [Wil20, p. 21]. Here, the isomorphism is 'rephrased' in terms of periods, which are interesting combinatorial objects connected to period polynomials. More information on this 'rephrasing' can be found in [CS17, Section 11.9] (this is actually the standard way in which the Eichler-Shimura isomorphism is presented).

Finally, we note that the Eichler-Shimura isomorphism and its variations are an active area of research. Indeed for example in [Nor19], the author discusses the Eichler-Shimura isomorphism for modular forms with respect to a congruence subgroups and discusses these in relation to a 'twisted L-function':

$$L(f \otimes e(x), s) = \sum_{n=1}^{\infty} \frac{a_n e(x)}{n^s},$$

for $e(x) = e^{2\pi i x}$ and $f(z) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(N))$, for some integer N. Here, $\mathcal{S}_k(\Gamma_0(N))$ are cusp forms with respect to the congruence subgroup $\Gamma_0(N)$, see [DS05, pp. 13–17] for details. This is similar to our definition of an L-function except with the coefficients rotated/twisted by $e(x)$. The author combines these twisted L-functions, the Eichler-Shimura isomorphism for congruence subgroups, period polynomials and probability to, in the authors words, 'determine the limiting distribution of the image of the

Eichler–Shimura map or equivalently the limiting joint distribution of the coefficients of the period polynomials associated to a fixed cusp form'.

Hopefully, all of the above demonstrates how fundamental the Eichler-Shimura isomorphism is and that it was a worthwhile result to conclude this project with.

Chapter 8

Conclusions

8.1 Summary

We began this project by first developing the basic theory of modular forms with respect to $SL_2(\mathbb{Z})$. This included defining and working with spaces of modular forms, Hecke operators and L-functions associated to cusp forms. Moreover, we have seen how these topics help us derive some very intriguing results related to number theory. Namely, we have shown Ramanujan's τ-function is multiplicative, we have given a not at all obvious relation between certain power divisor functions and we have shown that L-functions associated to Hecke eigenforms have an Euler product, similar to the Riemann zeta function. All of this gives merely a glimpse of how interesting and useful it can be to study modular forms.

In the final chapters we built up to proving the result know as the Eichler-Shimura isomorphism. This result allowed us to relate spaces of cusp forms to the first Eichler cohomology groups. Specifically, this isomorphism allowed us to relate cusp forms bijectively with classes of functions $\Gamma \to V_k(\Gamma)$. We have seen that this is a very intriguing result that used most of the work we have developed throughout this project and that it and its generalisations are still actively studied by mathematicians.

8.2 Further Work

Hopefully a reader is left excited about the study of modular forms and wants to look into the subject more deeply. There are many such avenues one could take. Here we have restricted ourselves to modular forms with respect to $SL_2(\mathbb{Z})$. One could begin by looking into modular forms with respect to congruence subgroups. These offer a more general setting to the one we have discussed here with similar objects such as Hecke operators, dimension formulae and L-functions. This route can lead one down the road towards modular curves, the modularity theorem and the proof of Fermat's last theorem, see [Wil95] and [DS05].

A different route a reader could take would be an extension of the Eichler-Shimura isomorphism as found in this project. For example one could follow on from what we have discussed in the previous chapter by reading [CS17, Section 11.9]. As mentioned before, this involves looking at periods and would prepare a reader for the usual way in which one would come across the Eichler-Shimura isomorphism in the literature.

Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

Acknowledgements

I am grateful to both Dr. Herbert Gangl for his enthusiasm, suggestions and advice throughout the year, as well as to my family for offering to proofread this project.

Bibliography

[Apo12] Tom M Apostol. *Modular Functions and Dirichlet Series in Number Theory*, volume 41. Springer Science & Business Media, 2012.

[Apo13] Tom M Apostol. *Introduction to Analytic Number Theory*. Springer Science & Business Media, 2013.

[Arm13] Mark Anthony Armstrong. *Basic topology*. Springer Science & Business Media, 2013.

[CS17] Henri Cohen and Fredrik Strömberg. *Modular Forms*, volume 179. American Mathematical Soc., 2017.

[DS05] Fred Diamond and Jerry Michael Shurman. *A First Course in Modular Forms*, volume 228. Springer, 2005.

[Duk99] William Duke. When is the product of two Hecke eigenforms an eigenform? *Number theory in progress*, 2:737–741, 1999.

[FJ02] D Farmer and Kevin James. The irreducibility of some level 1 Hecke polynomials. *Mathematics of Computation*, 71(239):1263–1270, 2002.

[GM12] Alexandru Ghitza and Angus McAndrew. Experimental evidence for Maeda's conjecture on modular forms. *Tbilisi Mathematical Journal*, 5(2):55–69, 2012.

[Gun62] Robert C Gunning. *Lectures on Modular Forms*. Number 48. Princeton University Press, 1962.

[Hat00] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, 2000.

[HM97] Haruzo Hida and Yoshitaka Maeda. Non-abelian base change for totally real fields. *pacific journal of mathematics*, 181(3):189–217, 1997.

[Ink20] Inkscape Project. *Inkscape version `1.1-dev`*, 2020. Available from `https://inkscape.org`.

[Jer06] Jonas Jermann. The valence formula and the space of modular forms, November 2006. `https://www2.math.ethz.ch/education/bachelor/seminars/`

ws0607/modular-forms/valence_formula_and_the_space_of_modular_forms.pdf.

[Kob12] Neal I Koblitz. *Introduction to Elliptic Curves and Modular Forms*, volume 97. Springer Science & Business Media, 2012.

[Mor17] Louis Joel Mordell. On Mr Ramanujan's empirical expansions of modular functions. In *Proc. Cambridge Philos. Soc*, volume 19, pages 117–124, 1917.

[Nor19] Asbjorn Christian Nordentoft. On the distribution of the Eichler-Shimura map and the zeroes of period polynomials. *arXiv preprint arXiv:1909.09622*, 2019.

[PAR19] PARI Group, Univ. Bordeaux. *PARI/GP version 2.12.0*, 2019. Available from http://pari.math.u-bordeaux.fr/.

[PT01] VV Prasolov and VM Tikhomirov. Geometry. Translations of Mathematical Monographs 200. Providence, RI. In *Am. Math. Soc*, 2001.

[Pyt20] Python Software Foundation. *Python version 3.8.2*, 2020. Available from https://www.python.org.

[Ram16] Srinivasa Ramanujan. On certain arithmetical functions. *Trans. Cambridge Philos. Soc*, 22(9):159–184, 1916.

[Sch17] Anthony J Scholl. Modular Forms Part III Lecture Notes, Lent 2017.

[Sch19] Anthony J Scholl. Modular Forms Part III Lecture Notes, Michaelmas 2019.

[SD73] Henry Peter Francis Swinnerton-Dyer. On ℓ-adic representations and congruences for coefficients of modular forms. In *Modular functions of one variable III*, pages 1–55. Springer, 1973.

[Ser12] Jean-Pierre Serre. *A Course in Arithmetic*, volume 7. Springer Science & Business Media, 2012.

[Was08] Lawrence C Washington. *Elliptic Curves: Number Theory and Cryptography.* CRC press, 2008.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Annals of mathematics*, 141(3):443–551, 1995.

[Wil20] Chris Williams. An Introduction to p-adic L-Functions II: Modular Forms, Spring 2020. https://warwick.ac.uk/fac/sci/maths/people/staff/cwilliams/lecturenotes/lecture_notes_part_ii.pdf.

[Zag00] Don Bernard Zagier. *Modular Forms of One Variable.* Instituto de Matemática Pura e Aplicada, 2000.

[Zag08] Don Zagier. Elliptic Modular Forms and Their Applications. In *The 1-2-3 of modular forms*, pages 1–103. Springer, 2008.

Appendix A

Python Code for Decomposing $SL_2(\mathbb{Z})$ Matrices

Here is the python code we have written to decompose a given matrix in $SL_2(\mathbb{Z})$ into a word in R and T following the algorithm followed in Theorem 1.2.2.

```
import numpy
from mpmath import *
mp.prec = 300


##creates the matrix R.
R=numpy.array([[0,-1],[1,0]])

##function computing a Mobius transformatimation.
def linearfrac(z,a,b,c,d):
    return mpc((a*z+b)/(c*z+d))

##function that gives the action of R on z
def multR(z):
    return linearfrac(z,0,-1,1,0)

##computes the decomposition of a matrix into a word in R and T.
def decomp(a,b,c,d):
    M=numpy.array([[a,b],[c,d]])
    l=[] ##list to count the number of Rs and Ts in decomp of M
    z=2j
    ##test vector, to check if our resulting matrix is I or -I.
    tv=numpy.array([[1],[0]])
    tv=numpy.dot(M,tv)
```

```
    z=linearfrac(z,M[0][0],M[0][1],M[1][0],M[1][1])
    ##loop that repeatedly uses R and T to bring M.z back to z.
    while not (abs(z-2j)<0.000000001):
        if (-0.5<z.real and z.real <0.5):
            z = multR(z)
            l.append(0)
            tv=numpy.dot(R,tv)
        else:
            intreal=round(z.real)
            l.append(int(intreal))
            z=z-intreal
            tv[0]=tv[0]+tv[1]*(-intreal)

    ## Checks whether the resulting matrix is I or -I.
    if (tv[0]==1):
        return(l)
    else:
        l.append(-1)
        return(l)

##output function
def wordinRandT(a,b,c,d):
    result=""
    l = decomp(a,b,c,d)
    if not l:
        print("[["+str(a)+","+str(b)+"]""["+str(c)+","+str(d)+"]]")
    else:
        for elmt in l:
            if elmt==0:
                result=result+"R^3"
            elif elmt==-1:
                result=result+"R^2"
            else:
                result=result+"T^"+str(elmt)
        print("[["+str(a)+","+str(b)+"]""["+str(c)+","+str(d)+"]]=")
        print(result)

wordinRandT(28,9,59,19)
```

Appendix B

Valence Formula Proof

Proof. Let C denote the anticlockwise contour around the boundary of $\mathcal{F}_R$ avoiding i, ρ, ρ^2 and any zeroes of f by going around around these points as described above. Under these assumptions we apply the argument principle to obtain

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{z\in\overline{\mathcal{F}_R}\setminus\{i,\rho,\rho^2\}} \mathrm{ord}_z(f) = \sum_{z\in\overline{\mathcal{F}}\setminus\{i,\rho,\rho^2\}} \mathrm{ord}_z(f).$$

Next we evaluate the integral explicitly. Let $\epsilon > 0$ be small and for $a \in \mathbb{H}$, $\phi,\ \theta \in [0, 2\pi)$, denote the curve of the circular arc $\gamma(t) = a + \epsilon e^{i(t+\theta)}$, for $0 \leq t \leq \phi$ by $C_{a,\epsilon,\theta,\phi}$. This curve is a circular arc with centre a, radius ϵ, beginning at an angle of θ above the horizontal which moves anticlockwise through an angle of ϕ.

i) The integrals along C_1 and C_3 cancel by Lemma 4.1.6, using $\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and that C_1 and C_3 have opposite orientations.

ii) The integral along C_2 becomes

$$\frac{1}{2\pi i}\int_{C_2} \frac{f'(z)}{f(z)}dz = -\int_{|q|=e^{-2\pi R}} \frac{\tilde{f}'(q)}{\tilde{f}(q)}dq = -\,\mathrm{ord}_\infty(f),$$

where we have used a change of variables $q = e^{2\pi iz}$ for the first equality and the argument principle for the last equality. The minus sign comes from the fact that the argument principle requires that you travel along the contour in an anticlockwise direction, whereas, when we change variables, noting the orientation of C_2, we will travel in a clockwise direction.

iii) Consider the Taylor series of f around a point $a \in \mathbb{H}$ with $\mathrm{ord}_a(f) = n$, $f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k = (z-a)^k g(z)$, where $g(z)$ is holomorphic in a small disc around a and $g(a) \neq 0$. Then we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{g'(z)}{g(z)} = \frac{n}{z-a} + h(z),$$

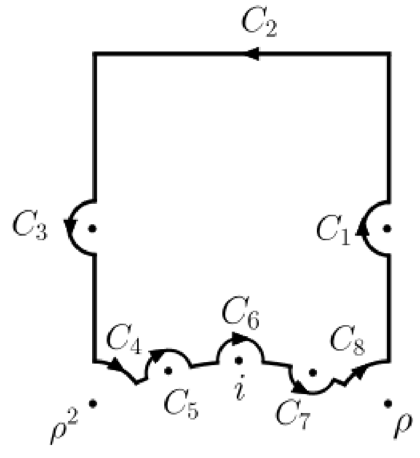

Figure 2.0: Diagram of the contour C.

where $h(z)$ is holomorphic in a small disc around a.

Integrating along C_6 and using the above,

$$-\frac{1}{2\pi i}\int_{C_{i,\epsilon,0,\pi}} \frac{f'(z)}{f(z)}dz = -\frac{1}{2\pi i}\int_{C_{i,\epsilon,0,\pi}} \frac{n}{z-i}dz - \frac{1}{2\pi i}\int_{C_{i,\epsilon,0,\pi}} h(z)dz.$$

The second integral on the right hand side vanishes as $\epsilon \to 0$. Calculating the first integral on the right hands side gives

$$-\frac{1}{2\pi i}\int_0^{\pi} \frac{n}{e^{i\phi}} ie^{i\phi} d\phi = -\frac{n}{2} = -\frac{\mathrm{ord}_i(f)}{2},$$

which holds for any $\epsilon > 0$

iv) Using exactly the same method as in (iii), noting the angle ϕ we are integrating over is $\frac{\pi}{3}$, we get the integrals along C_4 and C_8 evaluate to $-\frac{\mathrm{ord}_\rho(f)}{6}$ and $-\frac{\mathrm{ord}_{\rho^2}(f)}{6}$ respectively as $\epsilon \to 0$. However, since we have that $\mathrm{ord}_\rho(f) = \mathrm{ord}_{\rho^2}(f)$ (using Lemma 4.1.6 with $\gamma = R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$), we see that

$$\frac{1}{2\pi i}\int_{C_4} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C_8} \frac{f'(z)}{f(z)}dz = -\frac{\mathrm{ord}_\rho(f)}{3}.$$

v) Consider the sum of the integrals along C_5 and C_7. Note that $S \cdot C_5 = -C_7$, since R is an inversion composed with a reflection in the imaginary axis. Hence, using Lemma 4.1.6, we see that

$$\frac{1}{2\pi i}\int_{C_5} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C_7} \frac{f'(z)}{f(z)}dz = -\frac{k}{2\pi i}\int_{C_5} \frac{1}{z}dz \to \frac{k}{2\pi}\int_{C_{0,1,\frac{\pi}{2},\frac{2\pi}{3}}} \frac{1}{z}dt = \frac{k}{12}$$

in the limit as $\epsilon \to 0$.

Collecting all these terms together, we obtain the desired claim in the limit as $\epsilon \to 0$ $\square$

Appendix C

Proof of Proposition 4.1.19

Proof. Suppose $k \equiv 0 \pmod 4$. Thus, k is of the form $k = 12q + a$, where $a = 0, 4$ or 8. First note that for $0 \leq i < \dim \mathcal{M}_k(\Gamma) = \lfloor \frac{k}{12} \rfloor + 1$, we see that

$$k - 12i \geq k - 12 \left\lfloor \frac{k}{12} \right\rfloor = a \geq 0.$$

Let $k_n = 12n + 4r$, where $r = 0, 1$ or 2 and $n \in \mathbb{N}_0$. For $n = 0$, the claim holds by our previous observations. Now, assume the claim holds for some $n \geq 0$. Note, using Proposition 4.1.2, that $E_4^{k_{n+1}/4} \in \mathcal{M}_{k_{n+1}}(\Gamma)$ and is not a cusp form. By Lemma 4.1.16, we have that $\mathcal{S}_{k_{n+1}}(\Gamma) \cong \mathcal{M}_{k_n}(\Gamma)$ under the isomorphism

$$\phi : \mathcal{M}_{k_n} \to \mathcal{S}_{k_{n+1}}(\Gamma),\ f \mapsto \Delta f.$$

Hence, $S_{k_{n+1}}(\Gamma)$ has basis

$$\Delta\{\Delta^i E_4^{(k_n - 12i)/4} \mid 0 \leq i < \dim \mathcal{M}_{k_n}(\Gamma)\} = \{\Delta^{i+1} E_4^{(k_n-12i)/4} \mid 0 \leq i < \dim \mathcal{S}_{k_{n+1}}(\Gamma)\}.$$

Now, the set

$$\{\Delta^{i+1} E_4^{(k_n-12i)/4} \mid 0 \leq i < \dim \mathcal{S}_{k_{n+1}}(\Gamma)\} \cup \{E_4^{k_{n+1}/4}\}$$

$$= \{\Delta^i E_4^{(k_{n+1}-12i)/4} \mid 0 \leq i < \dim \mathcal{M}_{k_{n+1}}(\Gamma)\}$$

is linearly independent, since $E_4^{k_{n+1}/4}$ is not a cusp form. So, we have a linearly independent set of $\dim \mathcal{M}_{k_{n+1}}(\Gamma)$ elements of $\mathcal{M}_{k_{n+1}}(\Gamma)$. This means that this set is a basis of $\mathcal{M}_{k_{n+1}}(\Gamma)$. The claim for the case when k is divisible by four then follows by induction on n

Suppose $k \equiv 2 \pmod 4$. Firstly, k is of the form $k = 4n + 2$, for $n \in \mathbb{N}_0$. We see that $k - 12i - 6$ is divisible by 4, as $k - 12i - 6 = 4(n - 3i - 1)$. Furthermore, we have that $k - 12i - 6 \geq 0$, for $0 \leq i < \dim \mathcal{M}_k(\Gamma)$. To see this, suppose $k \not\equiv 2 \pmod{12}$ as well as

$k \equiv 2 \pmod 4$. Then, k is of the form $k = 12q + a$, where $a = 6$ or 10. So we have that

$$\begin{aligned} k - 12i - 6 &\geq k - 12(\dim \mathcal{M}_k(\Gamma) - 1) - 6 = k + 6 - 12 \dim \mathcal{M}_k(\Gamma) \\ &= k + 6 - 12\left(\left\lfloor \frac{k}{12} \right\rfloor + 1\right) = (12q + a) + 6 - 12q - 12 \\ &= a - 6 \geq 0. \end{aligned}$$

The proof of the case where $k \equiv 2 \pmod 4$ and $k \equiv 2 \pmod{12}$ follows exactly the same lines. The rest of the proof is identical to proof of the case $k \equiv 0 \pmod 4$. $\square$

Appendix D

Pari Code Used in Example 5.4.2

The following PARI/GP code calculates the q-expansions of the modular forms $\{\Delta^3, \Delta^2 E_4^3, \Delta E_4^6\}$ up to and including order six terms:

```
Ser(mfcoefs(mfDelta(),4))^3
%1 = x^3 - 72*x^4 + 2484*x^5 - 54528*x^6 + O(x^7)

Ser(mfcoefs(mfDelta(),5))^2*Ser(mfcoefs(mfEk(4),5))^3
%2 = x^2 + 672*x^3 + 145800*x^4 + 9111680*x^5 - 233907300*x^6
    + O(x^7)

Ser(mfcoefs(mfDelta(),6))*Ser(mfcoefs(mfEk(4),6))^6
%3 = x + 1416*x^2 + 842652*x^3 + 271386688*x^4 + 50558976510*x^5
    + 5356057835232*x^6 + O(x^7)
```

This next section of code calculates the characteristic polynomial of the endomorphism $T_2 : \mathcal{S}_{36}(\Gamma) \to \mathcal{S}_{36}(\Gamma)$ using the above basis:

```
charpoly([-6144,-331776000,-17915904000000;-72,144384,34629120000
            ;0,1,1416])
%4 = x^3 - 139656*x^2 - 59208339456*x - 1467625047588864
```

Finally, this last section of code checks whether the above characteristic polynomial is irreducible, calculates the discriminant of this polynomial and gives the prime factorisation of the aforementioned discriminant.

```
polisirreducible(x^3 - 139656*x^2 - 59208339456*x - 1467625047588864)
%5 = 1
```

```
poldisc(x^3 - 139656*x^2 - 59208339456*x - 1467625047588864)
%6 = 6060374850491967093448089010176600

factor(6060374850491967093448089010176600)
%8 =
[           2  30]

[           3  10]

[           5   2]

[           7   2]

[          23   1]

[        1259   1]

[269461929553   1]
```

COVID19 IMPACT SHEET
Project 3/4
Department of Mathematical Sciences

Student Name: Thomas Andrew Lamb
Year group (3/4): 4
Project Topic: Introduction to Modular Forms
Project supervisor(s): Dr. Herbert Gangl
Did Covid19 prevent you from completing part of your project report (Yes/No): No
If 'Yes', please indicate what it prevented you from doing (max 100 words): N/A
Please summarise the action taken in response (max 100 words): N/A