Polylogarithmic Integral Identities

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01/08/2019

1 List of Identities.

1.1 Polylogarithm Identities

For the first part of Section 2 we derive the following identities relating polylogarithms and multiple zeta values.

For m, n, k positive integers:

(a)

$$\int_0^1 \int_0^1 Li_2(xy) \frac{(1-y)^2}{(1-xy)^2} dxdy = -\zeta(3) + \frac{1}{2}\zeta(2) + \frac{1}{2}, \tag{1}$$

(b)

$$\int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)}{(1-xy)^2} dxdy$$
$$= \frac{\zeta(n)}{m+1} - \frac{1}{m(m+1)} \sum_{i=1}^{m} \sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + \frac{(-1)^{n-1}}{m(m+1)} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}}, \tag{2}$$

(c)

$$\int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)^2}{(1-xy)^2} dxdy$$
$$= \frac{\zeta(n)}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)} \sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{(m+1)^{j+1}}$$
$$- \frac{2}{m(m+1)(m+2)} \sum_{i=1}^{m} \sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{i^{j+1}}$$
$$+ (-1)^{n-1} \frac{2}{m(m+1)(m+2)} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}} + (-1)^n \frac{H_{m+1}}{(m+1)^n(m+2)}, \tag{3}$$

(d)

$$\int_0^1 \int_0^1 Li_n(xy)\frac{y^m(1-y)^{k+1}}{(1-xy)^2}dxdy$$
$$= \sum_{h=0}^{k}(-1)^h\binom{k}{h}\left[\frac{\zeta(n)}{m+h+1} - \frac{1}{(m+h)(m+h+1)}\sum_{i=1}^{m+h}\sum_{j=0}^{n-3}(-1)^j\frac{\zeta(n-1-j)}{i^{j+1}}\right.$$
$$\left. + \frac{(-1)^{n-1}}{(m+h)(m+h+1)}\sum_{i=1}^{m+h}\frac{H_i}{i^{n-1}}\right],$$

(e)

$$\zeta(n+1) = 2\int_0^1 \int_0^1 Li_n(xy)\frac{1-y}{x(1-xy)^2}dxdy,$$

where $H_i = \sum_{k=1}^{i}\frac{1}{k}$ the ith harmonic number and empty sums are interpreted as 0.

1.2 ψ_n Function Identities

In the Section 3 we define the function ψ_n function and derive the following identities relating the ψ_n function and derivatives of Multiple Zeta Values.

For m, n, k positive integers,

(a)

$$\int_0^1 \int_0^1 \frac{\psi_n(xy)}{xy-1}dxdy = \frac{d}{ds}\bigg|_{s=n}\zeta(2,s)$$

(b)

$$\int_0^1 \int_0^1 \psi_n(xy)\frac{x^{p-1}y^p}{1-xy}dxdy = \sum_{j=0}^{n-2}(-1)^{j+1}\frac{\zeta'(n-j)}{p^{j+1}} + (-1)^{n-1}\frac{\gamma_1 - \gamma_1(0,p)}{p^n}$$

(c)

$$\int_0^1 \int_0^1 \psi_n(xy)\frac{y(1-y)}{(1-xy)^2}dxdy = \frac{1}{2}\left(\sum_{j=0}^{n-2}(-1)^{j+1}\zeta'(n-j) + (-1)^{n-1}(\gamma_1 - \gamma_1(0,1))\right)$$

(d)

$$\int_0^1 \int_0^1 \psi_n(xy)\frac{y^m(y-1)}{(1-xy)^2}dxdy$$
$$= \frac{\zeta'(n)}{m+1} - \frac{1}{m(m+1)}\sum_{i=1}^{m}\sum_{j=0}^{n-3}(-1)^j\frac{\zeta'(n-1-j)}{i^{j+1}} + \frac{(-1)^n}{m(m+1)}\sum_{i=1}^{m}\frac{\gamma_1 - \gamma_1(0,i)}{i^{n-1}}$$

(e)

$$\zeta'(n+1) = 2\int_0^1 \int_0^1 \psi_n(xy)\frac{y-1}{x(1-xy)^2}dxdy$$

where empty sums are interpreted as 0.

2 Acknowledgements

In this paper we provide and prove several relations between polylogarithms and multiple zeta values. These were derived during the summer of 2019 under the supervision of Dr. Herbert Gangl. Previously, Gangl made a number of experimental observations using PARI/GP about potential relations between integrals of the form

$$\int_0^1 \int_0^1 Li_k(xy) \frac{y^l(1-y)^m}{(1-xy)^n}$$

and linear combinations of zeta and multiple zeta values; these crorrespond to equations 1, 2 and 3 labelled in Section 1.1 above. This paper has been influenced by Hoffman's work in [1]. Indeed, we will repeatedly use certain results from Hoffman's paper and follow its style. The goal of this paper to try an extend some of Hoffman's results as well as to attempt to prove some of Gangl's conjectural identities.

3 Polylogarithm Identities.

For m, n, k positive integers,

Proposition 1.

$$\int_0^1 \int_0^1 Li_2(xy) \frac{(1-y)^2}{(1-xy)^2} dxdy = -\zeta(3) + \frac{1}{2}\zeta(2) + 1.$$

Proof. Consider,

$$\begin{aligned}
\int_0^1 \int_0^1 Li_2(xy) \frac{(1-y)^2}{(1-xy)^2} dxdy &= \int_0^1 \int_0^1 Li_2(xy) \frac{1-y}{(1-xy)^2} dxdy - \int_0^1 \int_0^1 Li_2(xy) \frac{y(1-y)}{(1-xy)^2} dxdy \\
&= \zeta(2) - \zeta(2,1) - \frac{1}{2}(\zeta(2) - 1) \\
&= -\zeta(2,1) + \frac{1}{2}\zeta(2) + \frac{1}{2} \\
&= -\zeta(3) + \frac{1}{2}\zeta(2) + \frac{1}{2},
\end{aligned}$$

where for the second equality we have used Theorems 4 and 5 of [1], and for last inequality we have used Euler's identity. □

Lemma 1. *For positive integer n and non-negative integer m,*

$$\sum_{k,l \geq 1} \frac{1}{k^n(k+l+m)(k+l+m+1)} = \sum_{j=0}^{n-2} (-1)^j \frac{\zeta(n-j)}{(m+1)^{j+1}} + (-1)^{n-1} \frac{H_{m+1}}{(m+1)^n},$$

where $H_m = \sum_{i=1}^m \frac{1}{i}$, is the mth harmonic number and empty sums are interpreted as 0.

Proof. The left hand side reads,

$$\begin{aligned}
\sum_{k,l \geq 1} \frac{1}{k^n(k+l+m)(k+l+m+1)} &= \sum_{k \geq 1} \frac{1}{k^n} \sum_{l \geq 1} \left(\frac{1}{k+l+m} - \frac{1}{k+l+m+1} \right) \\
&= \sum_{k \geq 1} \frac{1}{k^n(k+m+1)},
\end{aligned}$$

using telescoping for the second inequality. The claim then follows from Lemma 1 of [1]. □

Lemma 2. *For positive integers m and $n \geq 2$,*

$$\sum_{k,l\geq 1}\frac{1}{k^{n-1}(k+l)(k+l+m)(k+l+m+1)}$$
$$=\frac{1}{m(m+1)}\sum_{i=1}^{m}\left(\sum_{j=0}^{n-3}(-1)^j\frac{\zeta(n-1-j)}{i^{j+1}}+(-1)^{n-2}\frac{H_i}{i^{n-1}}\right)$$
$$-\left(\sum_{j=0}^{n-3}(-1)^j\frac{\zeta(n-1-j)}{(m+1)^{j+2}}+(-1)^{n-2}\frac{H_{m+1}}{(m+1)^n}\right),$$

where $H_i=\sum_{k=1}^{i}\frac{1}{k}$, is the mth harmonic number and empty sums are interpreted as 0.

Proof. Splitting up the left hand side gives,

$$\sum_{k,l\geq 1}\frac{1}{k^{n-1}(k+l)(k+l+m)(k+l+m+1)}$$
$$=\sum_{k\geq 1}\frac{1}{k^{n-1}}\sum_{l\geq 1}\frac{1}{(k+l)(k+l+m)}-\sum_{k\geq 1}\frac{1}{k^{n-1}}\sum_{l\geq 1}\frac{1}{(k+l)(k+l+m+1)}.$$

The first term on the right hand side of the above equality becomes,

$$\sum_{k\geq 1}\frac{1}{k^{n-1}}\sum_{l\geq 1}\frac{1}{(k+l)(k+l+m)}=\frac{1}{m}\sum_{k\geq 1}\frac{1}{k^{n-1}}\sum_{l\geq 1}\left(\frac{1}{k+l}-\frac{1}{k+l+m}\right)$$
$$=\frac{1}{m}\sum_{k\geq 1}\left(\frac{1}{k^{n-1}(k+1)}+\frac{1}{k^{n-1}(k+2)}+...+\frac{1}{k^{n-1}(k+m)},\right)$$

using telescoping once again. Similarly, the second term of the right hand side of the initial equality becomes,

$$\sum_{k\geq 1}\frac{1}{k^{n-1}}\sum_{l\geq 1}\frac{1}{(k+l)(k+l+m+1)}$$
$$=\frac{1}{m+1}\sum_{k\geq 1}\frac{1}{k^{n-1}}\sum_{l\geq 1}\left(\frac{1}{k+l}-\frac{1}{k+l+m+1}\right)$$
$$=\frac{1}{m+1}\sum_{k\geq 1}\left(\frac{1}{k^{n-1}(k+1)}+\frac{1}{k^{n-1}(k+2)}+\cdots+\frac{1}{k^{n-1}(k+m)}+\frac{1}{k^{n-1}(k+m+1)}\right).$$

As a result,

$$\sum_{k,l\geq 1}\frac{1}{k^{n-1}(k+l)(k+l+m)(k+l+m+1)}$$
$$=\left(\frac{1}{m}-\frac{1}{m+1}\right)\sum_{k\geq 1}\sum_{i=1}^{m}\frac{1}{k^{n-1}(k+i)}-\frac{1}{m+1}\sum_{k\geq 1}\frac{1}{k^{n-1}(k+m+1)}$$

$$= \frac{1}{m(m+1)} \sum_{k\geq 1} \sum_{i=1}^{m} \frac{1}{k^{n-1}(k+i)} - \frac{1}{m+1} \sum_{k\geq 1} \frac{1}{k^{n-1}(k+m+1)}$$

$$= \frac{1}{m(m+1)} \sum_{i=1}^{m} \left(\sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + (-1)^{n-2} \frac{H_i}{i^{n-1}} \right)$$

$$- \left(\sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{(m+1)^{j+2}} + (-1)^{n-2} \frac{H_{m+1}}{(m+1)^n} \right),$$

using the standard approach of spiting into partial fractions, telescoping and Lemma 1 of [1] □

The following proves identity 2,

Theorem 1. *For positive integers m and n, with $n \geq 2$,*

$$\int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)}{(1-xy)^2} dxdy$$

$$= \frac{\zeta(n)}{m+1} - \frac{1}{m(m+1)} \sum_{i=1}^{m} \sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + \frac{(-1)^{n-1}}{m(m+1)} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}},$$

where $H_i = \sum_{k=1}^{i} \frac{1}{k}$ is the ith harmonic number and empty sums are interpreted as 0.

Proof. Beginning with the left hand side,

$$\int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)}{(1-xy)^2} dxdy = \int_0^1 \int_0^1 \sum_{k\geq 1} \sum_{i,j\geq 0} \frac{1}{k^n} x^{i+j+k} y^{i+j+k+m} (1-y) dxdy$$

$$= \sum_{k\geq 1} \sum_{i,j\geq 0} \frac{1}{k^n(k+i+j+1)} \left(\frac{1}{k+i+j+m+1} - \frac{1}{k+i+j+m+2} \right)$$

$$= \sum_{k,l\geq 1} \frac{l}{k^n(k+l)} \left(\frac{1}{k+l+m} - \frac{1}{k+l+m+1} \right)$$

$$= \sum_{k,l\geq 1} \frac{1}{k^{n-1}} \left(\frac{1}{k} - \frac{1}{k+l} \right) \left(\frac{1}{k+l+m} - \frac{1}{k+l+m+1} \right)$$

$$= \sum_{k,l\geq 1} \frac{1}{k^n(k+l+m)(k+l+m+1)} - \sum_{k,l\geq 1} \frac{1}{k^{n-1}(k+l)(k+l+m)(k+l+m+1)}$$

$$= \sum_{j=0}^{n-2} (-1)^j \frac{\zeta(n-j)}{(m+1)^{j+1}} + (-1)^{n-1} \frac{H_{m+1}}{(m+1)^n}$$

$$- \frac{1}{m(m+1)} \sum_{i=1}^{m} \left(\sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + (-1)^{n-2} \frac{H_i}{i^{n-1}} \right)$$

$$+ \left(\sum_{j=0}^{n-3} (-1)^j \frac{\zeta(n-1-j)}{(m+1)^{j+2}} + (-1)^{n-2} \frac{H_{m+1}}{(m+1)^n} \right)$$

$$= \sum_{j=0}^{n-2}(-1)^j \frac{\zeta(n-j)}{(m+1)^{j+1}} + \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{(m+1)^{j+2}}$$

$$- \frac{1}{m(m+1)} \sum_{i=1}^{m} \left(\sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + (-1)^{n-2} \frac{H_i}{i^{n-1}} \right)$$

$$= \frac{\zeta(n)}{m+1} - \frac{1}{m(m+1)} \sum_{i=1}^{m} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + \frac{(-1)^{n-1}}{m(m+1)} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}},$$

where we have used Lemmas (1) and (2). □

For the case when $m = 1$ in the above theorem, we see that the resulting expression agrees with that of Theorem 4 from [1].

Corollary 1. *For positive integers m and n, with $n \geq 2$,*

$$\int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)^2}{(1-xy)^2} dxdy$$

$$= \frac{\zeta(n)}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{(m+1)^{j+1}}$$

$$- \frac{2}{m(m+1)(m+2)} \sum_{i=1}^{m} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{i^{j+1}}$$

$$+ (-1)^{n-1} \frac{2}{m(m+1)(m+2)} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}} + (-1)^n \frac{H_{m+1}}{(m+1)^n(m+2)},$$

where $H_i = \sum_{k=1}^{i} \frac{1}{k}$ is the ith harmonic number and empty sums are interpreted as 0.

Proof. Expanding the left hand side and using Theorem (1) gives,

$$\int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)^2}{(1-xy)^2} dxdy = \int_0^1 \int_0^1 Li_n(xy) \frac{y^m(1-y)}{(1-xy)^2} dxdy - \int_0^1 \int_0^1 Li_n(xy) \frac{y^{m+1}(1-y)}{(1-xy)^2} dxdy$$

$$= \frac{\zeta(n)}{m+1} - \frac{1}{m(m+1)} \sum_{i=1}^{m} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} + \frac{(-1)^{n-1}}{m(m+1)} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}}$$

$$- \frac{\zeta(n)}{m+2} + \frac{1}{(m+1)(m+2)} \sum_{i=1}^{m+1} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} - \frac{(-1)^{n-1}}{(m+1)(m+2)} \sum_{i=1}^{m+1} \frac{H_i}{i^{n-1}}$$

$$= \frac{\zeta(n)}{(m+1)(m+2)} + (-1)^n \frac{H_{m+1}}{(m+1)^n(m+2)} + \frac{(-1)^{n-1}}{m+1} \sum_{i=1}^{m} \frac{H_i}{i^{n-1}} \left(\frac{1}{m} - \frac{1}{m+2} \right)$$

$$+ \frac{1}{(m+1)(m+2)} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{(m+1)^{j+1}} + \frac{1}{m+1} \sum_{i=1}^{m} \sum_{j=0}^{n-3}(-1)^j \frac{\zeta(n-1-j)}{i^{j+1}} \left(\frac{1}{m+2} - \frac{1}{m} \right)$$

$$= \frac{\zeta(n)}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)}\sum_{j=0}^{n-3}(-1)^j\frac{\zeta(n-1-j)}{(m+1)^{j+1}}$$
$$- \frac{2}{m(m+1)(m+2)}\sum_{i=1}^{m}\sum_{j=0}^{n-3}(-1)^j\frac{\zeta(n-1-j)}{i^{j+1}}$$
$$+ (-1)^{n-1}\frac{2}{m(m+1)(m+2)}\sum_{i=1}^{m}\frac{H_i}{i^{n-1}} + (-1)^n\frac{H_{m+1}}{(m+1)^n(m+2)}.$$

□

For $n = 2$, Corollary 1 reads,

$$\int_0^1\int_0^1 Li_2(xy)\frac{y^m(1-y)^2}{(1-xy)^2}dxdy = \frac{\zeta(2)}{(m+1)(m+2)} + \frac{H_{m+1}}{(m+1)^2(m+2)} - \frac{2}{m(m+1)(m+2)}\sum_{i=1}^{m}\frac{H_i}{i}.$$

Another corollary of Theorem 1 is identity 4.

Corollary 2. *For positive integers m and n, with $n \geq 2$ and non-negative integer k,*

$$\int_0^1\int_0^1 Li_n(xy)\frac{y^m(1-y)^{k+1}}{(1-xy)^2}dxdy$$
$$= \sum_{h=0}^{k}(-1)^h\binom{k}{h}\left[\frac{\zeta(n)}{m+h+1} - \frac{1}{(m+h)(m+h+1)}\sum_{i=1}^{m+h}\sum_{j=0}^{n-3}(-1)^j\frac{\zeta(n-1-j)}{i^{j+1}}\right.$$
$$\left.+\frac{(-1)^{n-1}}{(m+h)(m+h+1)}\sum_{i=1}^{m+h}\frac{H_i}{i^{n-1}}\right].$$

Proof. Manipulating the left hand side and using the Binomial Theorem gives,

$$\int_0^1\int_0^1 Li_n(xy)\frac{y^m(1-y)^{k+1}}{(1-xy)^2}dxdy = \int_0^1\int_0^1 Li_n(xy)\frac{y^m(1-y)}{(1-xy)^2}(1-y)^k dxdy =$$
$$= \int_0^1\int_0^1 Li_n(xy)\frac{y^m(1-y)}{(1-xy)^2}\sum_{h=0}^{k}\binom{k}{h}(-1)^h y^h dxdy$$
$$= \sum_{h=0}^{k}(-1)^h\binom{k}{h}\int_0^1\int_0^1 Li_n(xy)\frac{y^{m+h}(1-y)}{(1-xy)^2}dxdy.$$

The claim then follows from Theorem 1. □

Theorem 2. *For integer $n \geq 2$ we have,*

$$\zeta(n+1) = 2\int_0^1\int_0^1 Li_n(xy)\frac{1-y}{x(1-xy)^2}dxdy$$

Proof. We have,

$$\int_0^1\int_0^1 Li_n(xy)\frac{1-y}{x(1-xy)^2}dxdy = \int_0^1\int_0^1 \sum_{k\geq 1}\sum_{i,j\geq 0}\frac{x^{k+i+j-1}y^{k+i+k}}{k^n}(1-y)dxdy$$

$$= \sum_{k\geq 1}\sum_{i,j\geq 0}\frac{1}{k^n}\left(\frac{1}{(k+i+j)(k+i+j+1)}-\frac{1}{(k+i+j)(k+i+j+2)}\right)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0}\frac{l+1}{k^n(k+l)}\left(\frac{1}{k+l+1}-\frac{1}{k+l+2}\right)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0}\frac{l}{k^n(k+l)}\left(\frac{1}{k+l+1}-\frac{1}{k+l+2}\right)+\sum_{k\geq 1}\sum_{l\geq 0}\frac{1}{k^n(k+l)}\left(\frac{1}{k+l+1}-\frac{1}{k+l+2}\right)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0}\frac{1}{k^{n-1}}\left(\frac{1}{k}-\frac{1}{k+l}\right)\left(\frac{1}{k+l+1}-\frac{1}{k+l+2}\right)$$

$$+\sum_{k\geq 1}\sum_{l\geq 0}\frac{1}{k^n(k+l)}\left(\frac{1}{k+l+1}-\frac{1}{k+l+2}\right)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0}\frac{1}{k^n}\left[\left(\frac{1}{k+l+1}-\frac{1}{k+l+2}\right)+\left(\frac{1}{k+l}-\frac{1}{k+l+1}\right)-\frac{1}{2}\left(\frac{1}{k+l}-\frac{1}{k+l+2}\right)\right]$$

$$+\sum_{k\geq 1}\sum_{l\geq 0}\frac{1}{k^{n-1}}\left[\frac{1}{2}\left(\frac{1}{k+l}-\frac{1}{k+l+2}\right)-\left(\frac{1}{k+l}-\frac{1}{k+l+1}\right)\right]$$

$$= \frac{1}{2}\sum_{k\geq 1}\frac{1}{k^n}\left(\frac{1}{k+1}+\frac{1}{k}\right)+\frac{1}{2}\sum_{k\geq 1}\frac{1}{k^{n-1}}\left(\frac{1}{k+1}-\frac{1}{k}\right)$$

$$= \frac{1}{2}\sum_{j=0}^{n-2}(-1)^j\zeta(n-j)+\frac{(-1)^{n-1}}{2}+\frac{\zeta(n+1)}{2}+\frac{1}{2}\sum_{j=0}^{n-3}(-1)^j\zeta(n-1-j)+\frac{(-1)^{n-2}}{2}-\frac{\zeta(n)}{2}$$

$$= \frac{\zeta(n+1)}{2}-\frac{\zeta(n)}{2}+(-1)^{n-2}\frac{\zeta(2)}{2}+\frac{1}{2}\sum_{j=0}^{n-3}(-1)^j\left(\zeta(n-j)+\zeta(n-1-j)\right)$$

$$= \frac{\zeta(n+1)}{2}-\frac{\zeta(n)}{2}+(-1)^{n-2}\frac{\zeta(2)}{2}+\frac{\zeta(n)}{2}+(-1)^{n-3}\frac{\zeta(2)}{2}=\frac{\zeta(n+1)}{2}.$$

□

4 ψ_n Function Identities.

If we consider differentiating the polylogarithm with respect to its weight we get the following,

$$\frac{\mathrm{d}}{\mathrm{d}n}Li_n(z)=\frac{\mathrm{d}}{\mathrm{d}n}\sum_{k=1}^{\infty}\frac{z^k}{k^n}=-\sum_{k=1}^{\infty}\log(k)\frac{z^k}{k^n}$$

Using our previous results involving the Zeta Function, we can use the above to obtain some different integral identities involving derivatives of the Zeta function and Multiple Zeta Values. This motivates the following definition,

Definition 1. *We define,*

$$\psi_n(z) = \sum_{k=1}^{\infty} \log(k) \frac{z^k}{k^n}$$

First we want to know the radius of convergence of ψ_n.

Lemma 3. *$\psi_n(z)$ converges absolutely for $|z| < 1$.*

Proof. Let R denote the radius of convergence of ψ_n. Consider,

$$\lim_{k\to\infty} \left(\left| \left(\log(k+1) \frac{z^{k+1}}{(k+1)^n} \right) / \left(\log(k) \frac{z^n}{k^n} \right) \right| \right) =| z | \lim_{k\to\infty} \left(\left(\frac{k}{k+1} \right)^n \frac{\log(k+1)}{\log(k)} \right) =| z | .$$

Hence, by the ratio test $R = 1$. □

Consider the following, which will be useful in the proofs that will follow:

$$\frac{\mathrm{d}}{\mathrm{d}n} \zeta(n+m) = \frac{\mathrm{d}}{\mathrm{d}n} \sum_{k=1}^{\infty} \frac{1}{k^{n+m}} = -\sum_{k=1}^{\infty} \frac{\log(k)}{k^{n+m}} = \zeta'(n+m),$$

for an integer m.

The Stieltjes constants γ_k are defined by the Laurent expansion of ζ about 1. Similarly, one can define the generalised Stieltjes constants $\gamma_k(a)$ by the Laurent expansion of the Hurwitz zeta function about 1. An answer to a Maths Stack Exchange problem by Olivier Oloa [1] generalised this by defining 'poly-Stieltjes constants' by the Laurent expansion of the poly-Hurwitz zeta function about 0. Explicitly,

$$\gamma_k(a,b) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{\log^k(n+a)}{n+b} - \frac{\log^{k+1}(N)}{k+1} \right),$$

for $\mathrm{Re}(a), \mathrm{Re}(b) > -1$ and k a non-negative integer. See Oloa's answer for details. In particular, we will be using Theorem 2 from Oloa's answer to prove the following lemma.

Lemma 4. *For positive integers p and n,*

$$\sum_{k=1}^{\infty} \frac{\log(k)}{k^n(k+p)} = \sum_{j=0}^{n-2} (-1)^{j+1} \frac{\zeta'(n-j)}{p^{j+1}} + (-1)^{n-1} \frac{\gamma_1 - \gamma_1(0,p)}{p^n},$$

where γ_k refers to the Stieltjes constants, $\gamma_k(a,b)$ refers to the poly-Stieltjes constants and where empty sums are interpreted as 0.

Proof. For $n = 1$ we have,

$$\sum_{k=1}^{\infty} \frac{\log(k)}{k(k+p)} = \frac{\gamma_1(0,0) - \gamma_1(0,p)}{p} = \frac{\gamma_1 - \gamma_1(0,p)}{p}.$$

[1] https://math.stackexchange.com/questions/364452/evaluate-int-0-frac-pi2-frac11x21-tan-x-mathrm-dx

Assume the result holds for some $n \geq 1$. Then we have,

$$\sum_{k=1}^{\infty} \frac{\log(k)}{k^{n+1}(k+p)} = \frac{1}{p} \sum_{k=1}^{\infty} \left(\frac{log(k)}{k^{n+1}} - \frac{log(k)}{k^n(k+p)} \right)$$

$$= -\frac{\zeta'(n+1)}{p} - \sum_{j=0}^{n-2} (-1)^{j+1} \frac{\zeta'(n-j)}{p^{j+2}} + (-1)^n \frac{\gamma_1 - \gamma_1(0,p)}{p^{n+1}}$$

$$= \sum_{j=0}^{n-1} (-1)^{j+1} \frac{\zeta'(n+1-j)}{p^{j+1}} + (-1)^n \frac{\gamma_1 - \gamma_1(0,p)}{p^{n+1}}.$$

The result follows by induction on n. □

Using this result and replacing the polylogarithm function with our new ψ_n function we can follow a similar direction taken by Hoffman in [1] as well as develop some of the results from the previous section.

Proposition 2. *For positive integers n,*

$$\int_0^1 \int_0^1 \frac{\psi_n(xy)}{xy-1} dxdy = \frac{d}{ds}\Big|_{s=n} \zeta(2,s)$$

Proof. Beginning as usual by expanding the left hand side in series gives,

$$\int_0^1 \int_0^1 \frac{\psi_n(xy)}{1-xy} dxdy = \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \int_0^1 \int_0^1 \frac{\log(k) x^{k+i} y^{k+i}}{k^n} dxdy = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\log(k)}{k^n(k+l)^2} = -\frac{d}{ds}\Big|_{s=n} \zeta(2,s)$$

□

Proposition 3. *For positive integers n and p, we have*

$$\int_0^1 \int_0^1 \psi_n(xy) \frac{x^{p-1} y^p}{1-xy} dxdy = \sum_{j=0}^{n-2} (-1)^{j+1} \frac{\zeta'(n-j)}{p^{j+1}} + (-1)^{n-1} \frac{\gamma_1 - \gamma_1(0,p)}{p^n}$$

where empty sums are interpreted as 0.

Proof. The conclusion follows by following the proof from Hoffman [1], Theorem 3, noting the extra factor of $\log(k)$ and Lemma 4 above. □

Proposition 4. *For a positive integer n,*

$$\int_0^1 \int_0^1 \psi_n(xy) \frac{y(1-y)}{(1-xy)^2} dxdy = \frac{1}{2} \left(\sum_{j=0}^{n-2} (-1)^{j+1} \zeta'(n-j) + (-1)^{n-1} (\gamma_1 - \gamma_1(0,1)) \right)$$

where empty sums are interpreted as 0.

Proof. Once again we follow the proof in Hoffman [1] (of Theorem 4) and use Lemma 4. □

Theorem 3. *For a positive integer n,*

$$\int_0^1 \int_0^1 \psi_n(xy) \frac{y^m(y-1)}{(1-xy)^2} dxdy$$

$$= \frac{\zeta'(n)}{m+1} - \frac{1}{m(m+1)} \sum_{i=1}^{m} \sum_{j=0}^{n-3} (-1)^j \frac{\zeta'(n-1-j)}{i^{j+1}} + \frac{(-1)^n}{m(m+1)} \sum_{i=1}^{m} \frac{\gamma_1 - \gamma_1(0,i)}{i^{n-1}}$$

where empty sums are interpreted as 0.

Proof. We follow the proof of Theorem 1 outlined in Section 2 using Lemma 4 instead of Hoffman [1], Lemma 1. □

As you can see, we can derive similar results to everything that we derived in Section 2, replacing $Li_n(xy)$ with $\psi_n(xy)$.

Proposition 5. *For integer $n \geq 2$*

$$\zeta'(n+1) = 2\int_0^1 \int_0^1 \psi_n(xy)\frac{y-1}{x(1-xy)^2}dxdy$$

Proof. Differentiating the result of Theorem 2 with respect to n gives,

$$\frac{\mathrm{d}}{\mathrm{d}n}\int_0^1 \int_0^1 Li_n(xy)\frac{1-y}{x(1-xy)^2}dxdy = \int_0^1 \int_0^1 \psi_n(xy)\frac{y-1}{x(1-xy)^2}dxdy = \frac{\mathrm{d}}{\mathrm{d}n}\zeta(n+1) = \zeta'(n+1)$$

□

References

[1] Michael E Hoffman. *Polylogarithmic Integrals and MZVs*, 2018.

[2] Herbert Gangl *Conjectural formulas for double polylogarithmic integrals as MZVs [Private communication]*, 2019